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Stability groups of series in vector spaces

B.A.F. Wehrfritz

School of Mathematical Sciences, Queen Mary University of London, London E1 4NS, England, United Kingdom

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ABSTRACT

We study the relationships between the subsets of left and bounded left Engel elements and various canonical locally nilpotent subgroups, in particular the Hirsch–Plotkin radical and the Fitting subgroup, of the stability groups of series of subspaces in vector spaces over fields and division rings. In this we extend work of C. Casolo and O. Puglisi and of G. Traustason.

Introduction

Let V be a vector space over the field F and $\mathbf{L}$ a complete series (see [4] for definitions) of subspaces of V containing V and $\{0\}$. The stability group $S = \operatorname{Stab} \mathbf{L}$ of $\mathbf{L}$ in V is the set of all elements of $\mathrm{GL}(V)$ that normalize each subspace in $\mathbf{L}$ and centralize each jump of $\mathbf{L}$. Let $\mathrm{Fitt}(\mathbf{L})$ denote the set of all elements of S that stabilize some finite subseries of $\mathbf{L}$ containing V and $\{0\}$. Then $\mathrm{Fitt}(\mathbf{L})$ is always a subgroup of the Fitting subgroup $\mathrm{Fitt}(S)$ of S.

E-mail address: b.a.f.wehrfritz@qmul.ac.uk.

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In [1] Casolo and Puglisi prove that the Hirsch–Plotkin radical $\mathrm{HP}(S)$ of S is always unipotent and speculate that $\mathrm{HP}(S)$ is always equal to $\mathrm{Fitt}(\mathbf{L})$ and hence also to $\mathrm{Fitt}(S)$. Moreover, they prove this equality holds if $\mathbf{L}$ is an ascending series, or if V has countable dimension over F or if V has an $\mathbf{L}$-basis (see below for definition). In [6] Traustason proves that always $\mathrm{Fitt}(S) = \mathrm{Fitt}(\mathbf{L})$ and constructs an example showing that $\mathrm{HP}(S)$ is not always equal to $\mathrm{Fitt}(S)$ (and hence not always equal to $\mathrm{Fitt}(\mathbf{L})$).

Below we present alternative proofs of these results, proofs that also deliver four extensions. Firstly we show that in the three special cases mentioned above not only is $\mathrm{HP}(S)$ equal to $\mathrm{Fitt}(\mathbf{L})$, but so too is the set $L(S)$ of left Engel elements of S. Secondly we show that always the set $L^-(S)$ of bounded left elements of S is equal to $\mathrm{Fitt}(\mathbf{L})$. For any group G the union of all the normal subgroups of G that are hypercentral in their own right is a canonical subgroup of G that we denote here by $\mathrm{PH}(G)$. (PH stands for Philip Hall; the earliest reference to this subgroup, as far as I am aware, is in Hall's paper [3].) Clearly $\mathrm{Fitt}(G) \leq \mathrm{PH}(G) \leq \mathrm{HP}(G)$ for all groups G. Thirdly, here we prove that $\mathrm{PH}(S) = \mathrm{Fitt}(\mathbf{L})$, extending Traustason's result. Fourthly, and less interestingly, we replace fields by division rings; we also state our results for series rather than complete series.

Notation. Below V is a left vector space over the arbitrary division ring D (so $\mathrm{GL}(V)$ acts on V on the right; clearly we could just as well take right vector spaces) and $\mathbf{L} = \{(\Lambda_\alpha, V_\alpha) : \alpha \in \mathbf{A}\}$ is a series of subspaces of V covering $V \backslash \{0\}$ (e.g. see [4] Vol. 1 for definition and basic properties of series). If v is a non-zero element of V, there is a unique α in $\mathbf{A}$ with $v \in \Lambda_\alpha \backslash V_\alpha$. We just say that the jump $\Lambda_\alpha / V_\alpha$ covers v. Set $S = \mathrm{Stab}\,\mathbf{L}$; that is S is the set of all elements of $\mathrm{GL}(V)$ that normalize each subspace in $\mathbf{L}$ and centralize each jump of $\mathbf{L}$. Also let $\mathbf{L}^*$ denote the completion of $\mathbf{L}$, so $\mathbf{L}^*$ has the same jumps as $\mathbf{L}$ and hence $\mathrm{Stab}\,\mathbf{L}^* = S$. Let $\mathrm{Fitt}(\mathbf{L})$ denote the set of all elements g of S that stabilize some finite subseries of $\mathbf{L}^*$ containing $\{0\}$ and V. (Note the $\mathbf{L}^*$ here.) Finally an $\mathbf{L}$-basis of V is a basis $\mathbf{B}$ of V such that for each α in $\mathbf{A}$ the set $\mathbf{B} \cap (\Lambda_\alpha \backslash V_\alpha)$ is a basis of Λ_α modulo V_α meaning that $\Lambda_\alpha = (\bigoplus_b Db) \oplus V_\alpha$, where the b here runs over the set $\mathbf{B} \cap (\Lambda_\alpha \backslash V_\alpha)$. These do not always exist.

We prove the following.

Theorem A. *For all D, V, $\mathbf{L}$ and S as above, the following subsets of S are equal.*

a) *The set $L^-(S)$ of bounded left Engel elements of S.*
b) $\mathrm{Fitt}(\mathbf{L})$.
c) *The Fitting subgroup* $\mathrm{Fitt}(S)$ *of S.*
d) *The P. Hall subgroup* $\mathrm{PH}(S)$ *of S.*

Theorem B. *With D, V, L and S as above, if $\mathbf{L}$ is an ascending series, or if V has an $\mathbf{L}$-basis, or if V has countable dimension over F, then the following subsets of S are equal:*