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Complexity of modules over classical Lie superalgebras

Houssein El Turkey

Department of Mathematics and Physics, University of New Haven, 300 Boston Post Road, West Haven, CT 06516, United States

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ABSTRACT

The complexity of the simple and the Kac modules over the general linear Lie superalgebra $\mathfrak{gl}(m|n)$ of type A was computed by Boe, Kujawa, and Nakano in [2]. A natural continuation to their work is computing the complexity of the same family of modules over the ortho-symplectic Lie superalgebra $\mathfrak{osp}(2|2n)$ of type C. The two Lie superalgebras are both of *Type I* which will result in similar computations. In fact, our geometric interpretation of the complexity agrees with theirs. We also compute a categorical invariant, z-complexity, introduced in [2], and we interpret this invariant geometrically in terms of a specific detecting subsuperalgebra. In addition, we compute the complexity and the z-complexity of the simple modules over the *Type II* Lie superalgebras $\mathfrak{osp}(3|2)$, $D(2,1;\alpha)$, $G(3)$, and $F(4)$.

1. Introduction

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a classical Lie superalgebra (hence $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra) over the complex numbers, $\mathbb{C}$. Let $\mathcal{F} := \mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}$ be the category of finite-dimensional $\mathfrak{g}$-supermodules which are completely reducible over $\mathfrak{g}_{\bar{0}}$. The authors in [3] showed that

E-mail address: helturkey@newhaven.edu.

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$\mathcal{F}$ has enough projectives and it satisfies: (i) it is a self-injective category and (ii) every module in this category admits a projective resolution which has a polynomial rate of growth. For a module $M \in \mathcal{F}$, the complexity $c_{\mathcal{F}}(M)$ is the rate of growth of the minimal projective resolution of M.

In this paper we compute the complexity of the simple and the Kac modules for the orthosymplectic Lie superalgebra $\mathfrak{osp}(2|2n)$. Let $K(\lambda)$ (resp. $L(\lambda)$) be the Kac (resp. simple) module of highest weight λ. Let $\operatorname{atyp}(\lambda)$ denote the atypicality of λ (see Subsection 2.2). For $\mathfrak{osp}(2|2n)$, $\operatorname{atyp}(\lambda)$ is either zero or one. For typical λ (i.e. $\operatorname{atyp}(\lambda) = 0$), the simple and the Kac modules are projective and hence they have a zero complexity. For atypical λ (i.e. $\operatorname{atyp}(\lambda) = 1$), the complexity is computed in Theorems 3.2.1 and 3.5.1:

$$c_{\mathcal{F}}(L(\lambda)) = 2n + 1, \quad c_{\mathcal{F}}(K(\lambda)) = 2n.$$

These computations can be interpreted geometrically as follows. For a module M, let $\mathcal{X}_M$ denote the associated variety defined by Duflo and Serganova [7], and $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M)$ the support variety as defined in [5]. Then, if $X(\lambda)$ is a Kac or a simple module, we have the geometric interpretation of the complexity in Theorem 4.2.2:

$$c_{\mathcal{F}}(X(\lambda)) = \dim \mathcal{X}_{X(\lambda)} + \dim \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(X(\lambda)). \tag{1.0.1}$$

The authors in [2] introduced a categorical invariant called the z-complexity of modules and denoted it by $z_{\mathcal{F}}(-)$ (see [2, Section 9]). They computed the z-complexity of the simple and the Kac modules over $\mathfrak{gl}(m|n)$ and then used a detecting subsuperalgebra $\mathfrak{f}$ to interpret their computations geometrically. We carry these computations over to $\mathfrak{osp}(2|2n)$ and conclude in Theorems 5.1.1 and 5.2.1 that for an atypical λ, we have

$$z_{\mathcal{F}}(L(\lambda)) = 2, \quad z_{\mathcal{F}}(K(\lambda)) = 1. \tag{1.0.2}$$

Moreover, we show in Theorem 5.3.1 that if $X(\lambda)$ is a Kac or a simple module, we have

$$z_{\mathcal{F}}(X(\lambda)) = \dim \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(X(\lambda)). \tag{1.0.3}$$

The fact that our geometric interpretations of the complexity and the z-complexity agree with the results obtained in [2] was expected since both types A and C are *Type I* Lie superalgebras (Subsection 3.1). It was interesting to know if these interpretations would hold over *Type II* Lie superalgebras, hence we computed the complexity and the z-complexity of the simple (finite-dimensional) modules over $\mathfrak{osp}(3|2)$, and the three exceptional Lie superalgebras $D(2, 1; \alpha)$, $G(3)$, and $F(4)$. Our results show that equations (1.0.1) and (1.0.3) hold for the simple modules over these Lie superalgebras. The results in this paper raise the question of whether these geometric interpretations will hold over other classical Lie superalgebras, in particular types B and D.

The paper is organized as follows. In Section 2, we introduce the preliminaries for classical Lie superalgebras and their representations. We recall the definitions of atypicality, complexity, support variety, associated variety, and z-complexity of modules. In