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Orbit automata as a new tool to attack the order problem in automaton groups

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We introduce a new tool, called the orbit automaton, that describes the action of an automaton group G on the subtrees corresponding to the orbits of G on levels of the tree. The connection between G and the groups generated by the orbit automata is used to find elements of infinite order in certain automaton groups for which other methods failed to work.

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Introduction

Groups generated by automata were formally introduced in 1960s [1,2], but gained a significant attention after remarkable discoveries in 1970s and 1980s that the class of these groups contains counterexamples to several long-standing conjectures in group theory. The first such evidence came in 1972 with the construction by Aleshin of an infinite periodic group generated by two initial automata [3] (the complete proof can be found also in [4,5]). But the field truly started to thrive after works of Grigorchuk [5, 6] that introduced new methods of self-similarity and length contraction, and provided simpler counterexamples to the general Burnside problem, and the first counterexamples to the Milnor problem on growth in groups [7]. We will also mention work of Gupta and Sidki [8] that brought to life another series of related examples of infinite finitely generated p-groups and introduced a very powerful language of rooted trees to the field.

The class of automaton groups is particularly interesting from the computational viewpoint. The internal structure and complexity of these groups make computations by hands quite complicated, and sometimes infeasible. Even though the word problem is decidable for the whole class, other general algorithmic problems including the conjugacy problem and the isomorphism problem are known to be undecidable in general [9]. The order problem was recently shown to be undecidable in the classes of semigroups generated by automata [10] and groups generated by asynchronous automata [11]. However, the beauty of this class lies in the plethora of partial methods solving many algorithmic problems in majority of cases. For example, it was shown recently that the conjugacy problem and the order problem are decidable in the group of all, so-called, bounded automata [12].

Two software packages (`FR` [13] and `AutomGrp` [14]) for the `GAP` system [15] have been developed to address the computational demand in automaton groups and semigroups. Many of the partial methods implemented in these packages rely heavily on the contraction of the length of the words while one passes to the sections at the vertices of the tree on which the group acts. However, not all automaton groups possess this property. In particular, such contraction rarely happens in groups generated by the *reversible automata*. While working with these groups available software often fails to produce definite answers. At the same time, additional structure of reversible automata allows us to prove certain general results about the groups in this class. For example, in [16] it is proved that the finiteness problem is decidable in the class of groups generated by 2-state reversible automata, in [17] it is proved that infinite groups generated by connected 3-state reversible automata always contain elements of infinite order. Further, in [18] it is shown that invertible reversible automata that have no bireversible component with any number of states generate infinite torsion-free semigroups. There are also several papers [19–24] devoted to the realization of free groups and free products of groups as groups generated by automata. All of the automata in the constructed families are reversible and by now all known free non-abelian automaton groups are either generated by reversible automata, or build from such groups [25]. The above works use