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Cohomological periodicities of crystallographic groups

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We observe that an n-dimensional crystallographic group G has periodic cohomology in degrees greater than n if it contains a torsion free finite index normal subgroup $S \trianglelefteq G$ whose quotient G/S has periodic cohomology. We then consider a different type of periodicity. Namely, we provide hypotheses on a crystallographic group G that imply isomorphisms $H_i(G/\gamma_c T, \mathbb{F}) \cong H_i(G/\gamma_{c+d} T, \mathbb{F})$ for $\mathbb{F}$ the field of p elements and $\gamma_c T$ a term in the relative lower central series of the translation subgroup $T \leq G$. The latter periodicity provides a means of calculating the mod-p homology of certain infinite families of finite p-groups using a finite (machine) computation.

1. Introduction

An *n-dimensional crystallographic group* $G \leq \mathsf{Isom}(\mathbb{R}^n)$ is a discrete subgroup of the isometries of n-dimensional Euclidean space whose translations form a free abelian subgroup $T \leq G$ of dimension n. The *translation subgroup* T is a finite index normal subgroup of G and the quotient $P = G/T$ is called the *point group*. We describe two

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cohomological periodicities arising in the context of crystallographic groups; both provide a means of calculating infinite families of cohomology groups from finite computations.

To describe the first periodicity we say that a $\mathbb{Z}G$-resolution R_* involving boundary homomorphisms ∂_* is *periodic* of period d *in degrees greater than* m if there is equality of modules $R_i = R_{i+d}$ and boundary homomorphisms $\partial_{i+1} = \partial_{i+1+d}$ for all $i \geq m$. When $m = 0$ we simply say that such a resolution is *periodic*.

Proposition 1. *Let G be an n-dimensional crystallographic group with a torsion free normal subgroup $S \trianglelefteq G$ of finite index whose quotient $Q = G/S$ admits a periodic free $\mathbb{Z}Q$-resolution of $\mathbb{Z}$ of period d. Then G admits a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$ which is periodic of period d in degrees greater than n.*

Proposition 1 allows one to calculate the integral cohomology of certain crystallographic groups in all degrees by performing a computer computation of the integral cohomology in just the first few degrees. As an illustration consider the group $G = \mathrm{SpaceGroupBBNWZ}(2, 10)$ arising as the tenth group of dimension 2 in the Cryst [6] library of crystallographic groups available as part of the GAP [8] system for computational algebra. This group is generated by the four isometries

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+1 \\ y \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y+1 \end{pmatrix}.$$

The point group in this case is cyclic of order 4 and admits a periodic resolution of period 2. Using the homological algebra package HAP [7] for GAP to compute the integral homology of G in degrees up to and including 4, one obtains from Proposition 1 with $S = T$ that

$$H_i(G, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_4, & i = 1 \\ \mathbb{Z}, & i = 2 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4, & \text{odd } i \geq 3 \\ 0, & \text{even } i \geq 4. \end{cases}$$

In order to compute, say, $H_3(G, \mathbb{Z})$ in GAP one can use the following commands.

```
gap> G:=SpaceGroupBBNWZ(2,10);;
gap> GroupHomology(G,3);
[ 2, 4, 4 ]
```

The second type of periodicity concerns the *lower central series* of T relative to G, defined by setting $\gamma_1 T = T$ and $\gamma_{c+1} T = [\gamma_c T, G]$ for $c \geq 1$. There is an action of $P = G/T$ on the free abelian group $\gamma_c T$ given by conjugation, $P \times \gamma_c T \to \gamma_c T$, $(gT, a) \mapsto gag^{-1}$.