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Normality of orthogonal and symplectic nilpotent orbit closures in positive characteristic ☆

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ABSTRACT

In this note we investigate the normality of closures of orthogonal and symplectic nilpotent orbits in positive characteristic. We prove that the closure of such a nilpotent orbit is normal provided that neither type d nor type e minimal irreducible degeneration occurs in the closure, and conversely if the closure is normal, then any type e minimal irreducible degeneration does not occur in it. Here, the minimal irreducible degenerations of a nilpotent orbit are introduced by W. Hesselink in [7] (or see [11] from which we take Table 1 for the complete list of all minimal irreducible degenerations). Our result is a weak version in positive characteristic of [11, Theorem 16.2(ii)], one of the main results of [11] over complex numbers.

1. Preliminaries

1.1. Let G be a reductive algebraic group over an algebraically closed field $\mathbb{K}$ of positive characteristic, and $\mathfrak{g} = \mathrm{Lie}(G)$. A nilpotent orbit of G is an orbit of a nilpotent

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element in $\mathfrak{g}$ under the adjoint action of G. For all classical groups, the nilpotent orbits were parameterized in terms of partitions. The normality of closures of nilpotent orbit of classical group has been studied by several authors. However, there is still an open question to decide the normality of the closures of nilpotent orbits. Our purpose is to investigate such a problem.

1.2. In 1979 and in 1980s, Kraft and Procesi in [10] and [11] determined the normality of orbit closures for all complex classical groups by using smooth equivalent arguments (with few exceptions of the very even orbits in the special orthogonal group D_{2l} remaining, which was completed by Sommers in [14]). For other types, A. Broer in [1] finished the work on the normality of nilpotent orbit closures (corresponding to two pairwise orthogonal short roots) by vanishing result of cohomology of line bundles of flag variety.

In the case of positive-characteristic fields, J.F. Thomsen in [15] proved that A. Broer's result holds in good characteristic and decided the normality of some nilpotent orbit closures (corresponding to two pairwise orthogonal short roots). Generally, the difficulty of extending Kraft–Procesi method for linear general groups over complex numbers to the case of positive characteristic fields is the failure of the statement over $\mathbb{C}$ "*Assume the reductive algebraic group G acts regularly on an affine variety V. If $\pi : V \to V_0$ is a quotient map and $W \subset V$ is a G-stable subvariety, then the restriction of π to W is a quotient map onto $\pi(W)$.*" Donkin in [2] overcame this difficulty by means of representation theory. He viewed the coordinate ring $\mathbb{K}[V]$ as a G-module, then considered certain module filtration of $\mathbb{K}[V]$ (called good filtration). This enabled him to prove that all closures of nilpotent orbits of general linear groups in positive character are normal.

1.3. Recently, E. Goldstein in her doctoral thesis [4] investigated the normality of the closures of nilpotent orbits of orthogonal and symplectic groups. There Goldstein exploited Donkin's method to the orthogonal and symplectic groups in positive characteristic. She finally obtained Proposition 5.2 of [4] which is crucial to the present paper (see Theorem 2.1). With aid of Goldstein's theorem, we are able to decide the normality of some nilpotent orbit closures. Let us introduce our main result in the next subsections.

1.4. Throughout the paper, we always assume $\mathbb{K}$ is an algebraically closed filed of characteristic $p > 2$. Let V be finite dimensional vector space over $\mathbb{K}$, G be one of the algebraic groups $\mathrm{O}(V)$ or $\mathrm{Sp}(V)$ which is determined by a non-degenerate form $(\cdot,\cdot)$ with $(u, v) = \varepsilon(v, u)$ where $\varepsilon \in \{1, -1\}$. Call V a quadratic space of type ε (shortly an orthogonal space in case $\varepsilon = 1$, a symplectic space in case $\varepsilon = -1$). Let $\mathfrak{g} = \mathfrak{so}(V, \mathbb{K})$ or $\mathfrak{sp}(V, \mathbb{K})$ be the Lie algebra of G. Then the nilpotent orbits $\mathcal{O}_{\varepsilon,\sigma}$ under the adjoint action of G in $\mathfrak{g}$ are completely determined by partitions σ of $n = \dim(V)$. The corresponding Young diagram is called an ε-diagram. We denote by $|\sigma|$ the size of σ, which is equal to $\sum_{i=1}^{t} ir_i$ for $\sigma = [1^{r_1}2^{r_2}3^{r_3}\dots t^{r_t}]$.

There is a well-known classification result on nilpotent orbits of $\mathfrak{g} = \mathfrak{so}(V, \mathbb{K}), \mathfrak{sp}(V, \mathbb{K})$.