Journal of Algebra 443 (2015) 441–468

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Independence in computable algebra

Matthew Harrison-Trainor [a,*,1], Alexander Melnikov [b,2], Antonio Montalbán [a,c,3]

[a] *Group in Logic and the Methodology of Science, University of California, Berkeley, 94720, USA*
[b] *The Institute of Natural and Mathematical Sciences, Massey University, 0745, New Zealand*
[c] *Department of Mathematics, University of California, Berkeley, 94720, USA*

ARTICLE INFO

Article history:
Received 24 September 2014
Available online 22 June 2015
Communicated by Gerhard Hiss

MSC:
03D45
03C57
12Y05

Keywords:
Computable structure
Computable algebra
Pregeometry

ABSTRACT

We give a sufficient condition for an algebraic structure to have a computable presentation with a computable basis and a computable presentation with no computable basis. We apply the condition to differentially closed, real closed, and difference closed fields with the relevant notions of independence. To cover these classes of structures we introduce a new technique of *safe extensions* that was not necessary for the previously known results of this kind. We will then apply our techniques to derive new corollaries on the number of computable presentations of these structures. The condition also implies classical and new results on vector spaces, algebraically closed fields, torsion-free abelian groups and Archimedean ordered abelian groups.

* Corresponding author.
E-mail addresses: matthew.h-t@berkeley.edu (M. Harrison-Trainor), alexander.g.melnikov@gmail.com (A. Melnikov), antonio@math.berkeley.edu (A. Montalbán).

[1] Partially supported by the Berkeley Fellowship and NSERC grant PGSD3-454386-2014.
[2] Partially supported by the Packard Foundation.
[3] Partially supported by NSF grant DMS-1363310 and the Packard Fellowship.

http://dx.doi.org/10.1016/j.jalgebra.2015.06.004

1. Introduction

The main objects of this paper are computable algebraic structures. A countably infinite algebraic structure $\mathcal{A}$ is *computable* (Mal'cev [36] and Rabin [51]) if it admits a labeling of its domain by natural numbers so that the operations on $\mathcal{A}$ become Turing computable upon the respective labels. Such a numbering is called a *computable presentation*, a *computable copy*, or a *constructivization* of $\mathcal{A}$. Without loss of generality, we restrict ourselves to countable structures with domain ω (the natural numbers). Examples of computably presented structures include recursively presented groups with decidable word problem (Higman [26]) and explicitly presented fields (van der Waerden [58] and Fröhlich and Shepherdson [17]).

1.1. Independence with applications

Mal'cev and his mathematical descendants were perhaps the first to realize the fundamental role of various notions of independence in effective algebra, especially in the study of the number of computable presentations of structures. In his pioneering paper [37], Mal'cev made an important observation:

> The additive group $\mathbb{V}^\infty \cong \bigoplus_{i\in\mathbb{N}} \mathbb{Q}$ has two computable presentations that are not computably isomorphic.

This effect had never been seen before, since algorithms had mostly been applied to finitely generated structures whose presentations are effectively unique. Mal'cev noted that $\mathbb{V}^\infty$ clearly has a "good" computable presentation $\mathcal{G}$ that is built upon a computable basis, and he constructed a "bad" computable presentation $\mathcal{B}$ of $\mathbb{V}^\infty$ that has no computable basis. Clearly, $\mathcal{G}$ is not computably isomorphic to $\mathcal{B}$ (written $\mathcal{G} \ncong_{comp} \mathcal{B}$). Essentially the same argument applies to the algebraically closed field $\mathbb{U}$ of infinite transcendence degree [44]. Similarly, manipulations with bases were used in the study of the number of computable copies in the contexts of torsion-free abelian groups [13,47,23] and ordered abelian groups [21] of infinite rank (though for ordered abelian groups, the existence of a "bad" copy is a new result appearing in this paper). The latter two examples are nontrivial, since the existence of a "good" copy is *not* evident. Nonetheless, in all these examples the "good" copy $\mathcal{G}$ and the "bad" copy $\mathcal{B}$ are isomorphic *relative to* the halting problem, or Δ^0_2-isomorphic. Goncharov [23] showed that $\mathcal{G} \ncong_{comp} \mathcal{B}$ and $\mathcal{G} \cong_{\Delta^0_2} \mathcal{B}$ imply there exist *infinitely many* computable presentations of the structure up to computable isomorphism. Thus, in each case discussed above we get infinitely many effectively different presentations.

Notions of independence play a central role in the study of the combinatorial properties of effectively presented vector spaces and for other structures with an appropriate notion of independence. Such studies were quite popular in the 70s and 80s; the standard reference is the fundamental paper of Metakides and Nerode [43], see also [9–11,54,14] and,