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Centrally symmetric configurations of order polytopes

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ABSTRACT

It is shown that the toric ideal of the centrally symmetric configuration of the order polytope of a finite partially ordered set possesses a squarefree quadratic initial ideal. It then follows that the convex polytope arising from the centrally symmetric configuration of an order polytope is a normal Gorenstein Fano polytope.

Introduction

Gorenstein Fano polytopes (reflexive polytopes) are interested in many researchers since they correspond to Gorenstein Fano varieties and are related with mirror symmetry. See, e.g., [4, §8.3] and its references. One of the most important problem is to find new classes of Gorenstein Fano polytopes. The centrally symmetric configuration [9] of an

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integer matrix supplies one of the most powerful tools to construct normal Gorenstein Fano polytopes. The purpose of the present paper is to study the centrally symmetric configuration of the integer matrix associated with the order polytope of a finite partially ordered set.

Let $\mathbb{Z}^{d\times n}$ denote the set of $d\times n$ integer matrices. Given $A \in \mathbb{Z}^{d\times n}$ for which no column vector is a zero vector, the *centrally symmetric configuration* of A is the $(d+1)\times(2n+1)$ integer matrix

$$A^{\pm} = \left[\begin{array}{c|ccc|ccc} 0 & & & & & & \\ \vdots & & A & & & -A & \\ 0 & & & & & & \\ \hline 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \end{array}\right].$$

On the other hand, the *centrally symmetric polytope* arising from A is the convex polytope $\mathcal{Q}_A^{(\mathrm{sym})}$ which is the convex hull in $\mathbb{R}^d$ of the column vectors of the matrix

$$\left[\begin{array}{c|c|c} 0 & & \\ \vdots & A & -A \\ 0 & & \end{array}\right].$$

We focus our attention on the problem when $\mathcal{Q}_A^{(\mathrm{sym})}$ is a normal Gorenstein Fano polytope. In general, the origin is contained in the interior of $\mathcal{Q}_A^{(\mathrm{sym})} \subset \mathbb{R}^d$. Suppose that $A \in \mathbb{Z}^{d\times n}$ satisfies $\mathbb{Z}A = \mathbb{Z}^d$. (Here $\mathbb{Z}A = \{z_1\mathbf{a}_1 + \cdots + z_n\mathbf{a}_n \mid z_1, \ldots, z_n \in \mathbb{Z}\}$ for $A = [\mathbf{a}_1, \ldots, \mathbf{a}_n] \in \mathbb{Z}^{d\times n}$.) Then Lemma 1.1 in Section 1 guarantees that if the toric ideal $I_{A^{\pm}}$ of $A^{\pm}$ possesses a squarefree initial ideal with respect to a reverse lexicographic order for which the variable corresponding to the column $[0, \ldots, 0, 1]^t$ is smallest, then $\mathcal{Q}_A^{(\mathrm{sym})}$ is a normal Gorenstein Fano polytope.

In [9], it is shown that if $\mathrm{rank}(A) = d$ and all nonzero maximal minors of A are ± 1, then $\mathcal{Q}_A^{(\mathrm{sym})}$ is a normal Gorenstein Fano polytope. However, the converse does not hold. It is mentioned the existence of a matrix A such that A does not satisfy the above condition but $\mathcal{Q}_A^{(\mathrm{sym})}$ is a normal Gorenstein Fano polytope [9, Example 2.16]. Hence to construct an infinite family of matrices A like this is an important problem. The aim of this paper is to give an answer of this problem by using partially ordered sets.

Let P be a partially ordered set (poset) on $[d] = \{1, \ldots, d\}$ and $\mathbf{e}_1, \ldots, \mathbf{e}_d$ the unit coordinate vectors of $\mathbb{R}^d$. Given a subset $\alpha \subset P$, we write $\rho(\alpha) \in \mathbb{R}^d$ for the vector $\sum_{i\in\alpha} \mathbf{e}_i$. A *poset ideal* of P is a subset $\alpha \subset P$ such that if $a \in \alpha$ and $b \in P$ together with $b \leq a$, then $b \in \alpha$. In particular, the empty set as well as P itself is a poset ideal of P. Let $\mathcal{J}(P)$ denote the set of poset ideals of P. Note that $\mathcal{J}(P)$ has the structure of a distributive lattice under inclusion, moreover, for any distributive lattice D, there exists a poset P such that $D \cong \mathcal{J}(P)$ by Birkhoff's Theorem [2]. The *order polytope* $\mathcal{O}(P)$ is the d-dimensional polytope which is the convex hull of $\{\rho(\alpha) \mid \alpha \in \mathcal{J}(P)\}$ in $\mathbb{R}^d$. See [11].