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Riemann surfaces with maximal real symmetry

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A R T I C L E I N F O

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A B S T R A C T

Let S be a compact Riemann surface of genus $g > 1$, and let $\tau : S \to S$ be any anti-conformal automorphism of S, of order 2. Such an anti-conformal involution is known as a *symmetry* of S, and the species of all conjugacy classes of all symmetries of S constitute what is known as the *symmetry type* of S. The surface S is said to have *maximal real symmetry* if it admits a symmetry $\tau : S \to S$ such that the compact Klein surface S/τ has maximal symmetry (which means that S/τ has the largest possible number of automorphisms with respect to its genus). If τ has fixed points, which is the only case we consider here, then the maximum number of automorphisms of S/τ is $12(g-1)$. In the first part of this paper, we develop a computational procedure to compute the symmetry type of every Riemann surface of genus g with maximal real symmetry, for given small values of $g > 1$. We have used this to find all of them for $1 < g \leq 101$, and give details for $1 < g \leq 25$ (in an appendix). In the second part, we determine the symmetry types of four infinite families of Riemann surfaces with maximal real symmetry. We also

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determine the full automorphism group of the Klein surface S/τ associated with each symmetry $\tau : S \to S$.

1. Introduction

Let S be a compact Riemann surface of genus $g > 1$, and let $\mathrm{Aut}^+(S)$ be the group of all conformal automorphisms of S, and $\mathrm{Aut}(S)$ be the full automorphism group of S, including both conformal and anti-conformal automorphisms (when the latter exist). An anti-conformal automorphism $\tau : S \to S$ of order 2 is known as a *symmetry* of S.

We may associate with each such τ a quantity known as the *species* of τ, defined as follows. Let k be the number of connected components (or *ovals*) of the fixed-point set $\mathrm{Fix}(\tau)$ of τ, and define $\varepsilon = +1$ if the orbit space S/τ of S under the action of $\langle\tau\rangle$ is orientable (or equivalently, if $S - \mathrm{Fix}(\tau)$ is not connected), and $\varepsilon = -1$ otherwise. Then the species of τ, denoted by $\mathrm{spc}(\tau)$, is given by $\mathrm{spc}(\tau) = \varepsilon k$. It is known that $\mathrm{spc}(\tau)$ determines τ up to homeomorphism. In particular, every conjugate of τ in the group $\mathrm{Aut}(S)$ has the same species as τ.

The *symmetry type* of S is the unordered list of species of representatives of all conjugacy classes of symmetries of S. This concept was introduced in [8].

There are very few families of Riemann surfaces for which the symmetry types are known. In this paper we address this issue, for a particular class of Riemann surfaces, namely those with maximal real symmetry.

If τ is any symmetry of the compact Riemann surface S, then the orbit space S/τ endowed with the dianalytic structure inherited naturally from S is known as a *Klein surface*. The *algebraic genus* of S/τ is defined to be the genus of S. Details are given in [1], where it is also shown that if S has genus $g > 1$, then since $\mathrm{Aut}(S)$ is finite, the same is true of the group $\mathrm{Aut}(S/\tau)$ of all automorphisms of S/τ, because the latter can be identified with the group of all conformal automorphisms of S that commute with τ (or in other words, the centraliser of τ in $\mathrm{Aut}^+(S)$).

A compact Riemann surface S of genus $g > 1$ is said to have *maximal real symmetry* if it admits a symmetry $\tau : S \to S$ such that the compact Klein surface S/τ has maximal symmetry (which means that S/τ has the largest possible number of automorphisms with respect to its genus). If S/τ has non-empty boundary, which is the only case we will consider here, then this maximum number is $12(g-1)$; see [15]. The automorphism groups of such bordered surfaces are called *M^*-groups*. These groups are smooth quotients of the extended modular group $\mathrm{PGL}(2,\mathbb{Z})$, and play a role for compact bordered Klein surfaces analogous to the one played by Hurwitz groups (smooth quotients of the ordinary $(2,3,7)$ triangle group) for compact Riemann surfaces. In contrast, however, relatively little is known about M^*-groups.

The contents of this paper can be summarised as follows. We give some further background in Section 2, and then in Section 3 we describe the structure of the full group