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On equivariant quantum Schubert calculus for G/P

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ABSTRACT

We show a $\mathbb{Z}^2$-filtered algebraic structure and a "quantum to classical" principle on the torus-equivariant quantum cohomology of a complete flag variety of general Lie type, generalizing earlier works of Leung and the second author. We also provide various applications on equivariant quantum Schubert calculus, including an equivariant quantum Pieri rule for partial flag variety $F\ell_{n_1,\cdots,n_k;n+1}$ of Lie type A.

1. Introduction

The complex Grassmannian $Gr(m, n+1)$ parameterizes m-dimensional complex vector subspaces of $\mathbb{C}^{n+1}$. The integral cohomology ring $H^*(Gr(m, n+1), \mathbb{Z})$ has an additive basis of Schubert classes σ^ν, indexed by partitions $\nu = (\nu_1, \cdots, \nu_m)$ inside an $m \times (n+1-m)$ rectangle: $n+1-m \geq \nu_1 \geq \cdots \geq \nu_m \geq 0$. The initial classical Schubert calculus, in modern language, refers to the study of the ring structure of $H^*(Gr(m, n+1), \mathbb{Z})$. The content includes

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(1) a Pieri rule, giving a combinatorial formula for the cup product by a set of generators of the cohomology ring, for instance by the special Schubert classes σ^{1^p}, $1 \le p \le m$, where $1^p = (1, \cdots, 1, 0, \cdots, 0)$ has precisely p copies of 1;
(2) more generally, a Littlewood–Richardson rule, giving a (manifestly positive) combinatorial formula of the structure constants $N_{\mu,\nu}^{\eta}$ in the cup product $\sigma^\mu \cup \sigma^\nu = \sum_\eta N_{\mu,\nu}^{\eta}\sigma^\eta$;
(3) a ring presentation of $H^*(Gr(m, n+1), \mathbb{Z})$;
(4) a Giambelli formula, expressing every σ^ν as a polynomial in special Schubert classes.

The complex Grassmannian $Gr(m, n+1)$ is a special case of homogeneous varieties G/P, where G denotes the adjoint group of a complex simple Lie algebra of rank n, and P denotes a parabolic subgroup of G. The classical Schubert calculus, in general, refers to the study of the classical cohomology ring $H^*(G/P) = H^*(G/P, \mathbb{Z})$.

There are various extensions of the classical Schubert calculus by replacing "classical" with "equivariant", "quantum", or "equivariant quantum". The equivariant quantum Schubert calculus for G/P refers to the study of the (integral) torus-equivariant quantum cohomology ring $QH_T^*(G/P)$, which is a deformation of the ring structure of the torus-equivariant cohomology $H_T^*(G/P)$ by incorporating genus zero, three-point equivariant Gromov–Witten invariants. In analogy with $H^*(Gr(m, n+1))$, the ring $QH_T^*(G/P)$ has a basis of Schubert classes σ^u over $H_T^*(\mathrm{pt})[q_1, \cdots, q_k]$ where $k := \dim H_2(G/P)$, indexed by elements in a subset W^P of the Weyl group W of G. The structure coefficients $N_{u,v}^{w,\mathbf{d}}$ of the equivariant quantum product

$$\sigma^u \star \sigma^v = \sum_{w\in W^P, \mathbf{d}\in H_2(G/P,\mathbb{Z})} N_{u,v}^{w,\mathbf{d}} \sigma^w q^{\mathbf{d}}$$

are homogeneous polynomials in $H_T^*(\mathrm{pt}) = \mathbb{Z}[\alpha_1, \cdots, \alpha_n]$ with variables α_i being simple roots of G. They contain all the information in the former kinds of Schubert calculus. For instance, the non-equivariant limit of $N_{u,v}^{w,\mathbf{d}}$, given by evaluating $(\alpha_1, \cdots, \alpha_n) = \mathbf{0}$, recovers an ordinary Gromov–Witten invariant, which counts the number of degree $\mathbf{d}$ rational curves in G/P passing through three Schubert subvarieties associated to u, v, w.

When $G = PSL(n+1, \mathbb{C})$, the Weyl group W is a permutation group S_{n+1} generated by transpositions $s_i = (i, i+1)$. Every homogeneous variety $PSL(n+1, \mathbb{C})/P$ is of the form $F\ell_{n_1,\cdots,n_k;n+1} := \{V_{n_1} \leqslant \cdots \leqslant V_{n_k} \leqslant \mathbb{C}^{n+1} \mid \dim_{\mathbb{C}} V_{n_j} = n_j, \ \forall 1 \le j \le k\}$, parameterizing partial flags in $\mathbb{C}^{n+1}$. As an algebra over $H_T^*(\mathrm{pt})[q_1, \cdots, q_k]$, the equivariant quantum cohomology ring $QH_T^*(F\ell_{n_1,\cdots,n_k;n+1})$ is generated (see e.g. [1,31]) by special Schubert classes

$$\sigma^{c[n_i,p]}, \quad \text{where} \quad c[n_i, p] := s_{n_i-p+1} \cdots s_{n_i-1} s_{n_i}.$$

One of the main results of our present paper is the following equivariant quantum Pieri rule. The non-equivariant limit of it recovers the quantum Pieri rule, which was first given by Ciocan-Fontanine [14], and was reproved by Buch [5]. The classical limit (by