Journal of Algebra 441 (2015) 84–107

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Intersections via resolutions

Joseph Ross

University of Southern California, Mathematics Department, 3620 South Vermont Avenue, Los Angeles, CA 90089, United States

A R T I C L E I N F O

Article history:
Received 25 April 2014
Available online 25 August 2015
Communicated by V. Srinivas

Keywords:
Algebraic cycles
Intersection theory
Resolution of singularities
Intersection homology

A B S T R A C T

We investigate the viability of defining an intersection product on algebraic cycles on a singular algebraic variety by pushing forward intersection products formed on a resolution of singularities. For varieties with resolutions having a certain structure (including all varieties over a field of characteristic zero), we obtain a stratification which reflects the geometry of the centers and the exceptional divisors. This stratification is sufficiently fine that divisors can be intersected with r-cycles (for $r \geq 1$), and 2-cycles can be intersected on a fourfold, provided their incidences with the strata are controlled. Similar pairings are defined on a variety with one-dimensional singular locus.

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E-mail address: joseph.ross@gmail.com.

http://dx.doi.org/10.1016/j.jalgebra.2015.06.025

1. Introduction

The (Borel–Moore) homology of a smooth manifold possesses a canonical ring structure in which the product is represented by intersection of cycle classes. The Chow groups of a smooth algebraic variety over the complex numbers also admit a ring structure, and the intersection products are compatible via the cycle class map [6, Cor. 19.2]. In both contexts, a well-behaved intersection theory for cycles fails to extend to spaces with singularities.

In topology, this motivated the definition of cohomology and its cup product operation, which in some sense isolates a subset of the space of cycles (with a different equivalence relation) which can be intersected even if the ambient space has singularities. There are several analogues in algebraic geometry. The Friedlander–Lawson theory of algebraic cocycles [4] is a geometric approach built from finite correspondences to projective spaces, and similar constructions underlie the motivic cohomology of Friedlander–Voevodsky [5]. The operational Chow cohomology of Fulton [6], on the other hand, adopts a more formal approach.

The intersection homology groups of Goresky and MacPherson [7] provide an interpolation between cohomology and homology: at least for a normal space X there is a sequence of groups:

$$H^{\dim X-*}(X) = IH_*^{\bar{0}}(X) \to \cdots \to IH_*^{\bar{p}}(X) \to \cdots \to IH_*^{\bar{t}}(X) = H_*(X)$$

factoring the cap product map. The decoration $\bar{p}$, called the perversity, is a sequence of integers which prescribes how cycles may meet the strata in a suitable stratification of the (possibly singular) space X; in the display above it increases from left to right. Each intersection homology group $IH_r^{\bar{p}}(X)$ arises as the homology of a complex of chains (either simplicial chains with respect to a triangulation [7], or singular chains [9]) of perversity $\bar{p}$.

One of the most interesting features of this theory is the existence of intersection pairings

$$IH_r^{\bar{p}}(X) \otimes IH_s^{\bar{q}}(X) \to IH_{r+s-\dim(X)}^{\bar{p}+\bar{q}}(X)$$

(provided $\bar{p} + \bar{q} \leq \bar{t}$) generalizing the cap product pairing between cohomology and homology, and providing a generalization of Poincaré duality to singular spaces.

The author and Eric Friedlander [3] have defined an algebraic cycle counterpart to the geometric approach of Goresky–MacPherson. In particular we have defined perverse Borel–Moore motivic homology groups $H_m^{\bar{p}}(X, \mathbb{Z}(r))$ (for a stratified variety X and a perversity $\bar{p}$) with a cycle class map to the Goresky–MacPherson theory in Chow degree: