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Transitive representations of inverse semigroups

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ABSTRACT

While every group is isomorphic to a transitive group of permutations, the analogous property fails for inverse semigroups: not all inverse semigroups are isomorphic to transitive inverse semigroups of one-to-one partial transformations of a set. We describe those inverse semigroups that are.

1. Introduction and preliminaries

This paper consists of three sections and an afterword. Section 1 explains a problem that was raised by Wagner between 1948 and 1952 and is solved in this paper. Section 2 explains non-algebraic heuristic ideas that led to the solution. Section 3 contains the main result and all necessary proofs.

We begin with definitions that make this paper reasonably self-contained. Different categories of readers may be more or less familiar with some of them.

A *semigroup* is a nonempty set with associative multiplication. If S is a semigroup and $sts = s$, $tst = t$ for $s, t \in S$, then t is called an *inverse* for s. A semigroup is *regular*

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if each of its elements has an inverse. A semigroup is *inverse* if each of its elements has a *uniquely determined* inverse. If S is inverse and $s \in S$, then s^{-1} denotes the unique inverse of s. Alternatively, inverse semigroups are precisely regular semigroups with commuting idempotent elements. A nonempty subset T of an inverse semigroup S is called an *inverse subsemigroup* of S if T is closed under multiplication and inversion, that is, $(\forall s,t)[s,t \in T \Rightarrow st \in T]$ and $(\forall s)[s \in T \Rightarrow s^{-1} \in T]$.

A *partial transformation* of a set A is a mapping φ of a subset B of A into A. Thus $\varphi(b)$ exists for all $b \in B$ and is not defined for $b \notin B$. We will write $b\varphi$ rather than $\varphi(b)$. We say that B is the *first projection* (or the *domain*) of φ and write $B = pr_1\varphi$. We say that φ is *one-to-one* if $a\varphi = b\varphi \Rightarrow a = b$ for all $a, b \in B$. Let $\mathcal{I}_A$ denote the set of all one-to-one partial transformations of A. It is clear (and well known) that $\mathcal{I}_A$ is closed under the usual composition of partial transformations: $\varphi, \psi \in \mathcal{I}_A \Rightarrow \varphi \circ \psi \in \mathcal{I}_A$. Here we may omit $\circ$ and write merely $\varphi\psi$. Then $a(\varphi\psi) = (a\varphi)\psi$ for every $a \in A$ such that both $a\varphi$ and $(a\varphi)\psi$ are defined, that is, $a \in pr_1\varphi$ and $a\varphi = \varphi(a) \in pr_1\psi$. In particular, $\varphi\psi$ may be the empty partial transformation $\varnothing$. This empty partial transformation is obviously one-to-one and hence belongs to $\mathcal{I}_A$.[1] Thus $\mathcal{I}_A$ is a semigroup of partial transformations. Also, $\varphi \in \mathcal{I}_A \Rightarrow \varphi^{-1} \in \mathcal{I}_A$. Here φ^{-1} is the *inverse* transformation for φ, that is, $(\forall a, b \in A)[a\varphi^{-1} = b \Leftrightarrow b\varphi = a]$. Clearly, φ^{-1} is the inverse for φ in the sense of the theory of inverse semigroups because $\varphi\varphi^{-1}\varphi = \varphi$ and $\varphi^{-1}\varphi\varphi^{-1} = \varphi^{-1}$ for every $\varphi \in \mathcal{I}_A$. We call $\mathcal{I}_A$ the *symmetric inverse semigroup on A*. Inverse subsemigroups of $\mathcal{I}_A$ are called *inverse semigroups of one-to-one partial transformations* of A.

Definition 1. An inverse semigroup Φ of one-to-one partial transformations of a set A is called *transitive* if, for every $a, b \in A$, there exists $\varphi \in \Phi$ such that $a\varphi = b$. An inverse semigroup is called *noble*[2] if it is isomorphic to a transitive inverse semigroup of one-to-one partial transformations of a set. Observe that a trivial group and a two-element semilattice are both noble because they are isomorphic to transitive inverse subsemigroups of $\mathcal{I}(A)$, where A is a singleton $\{a\}$. We exclude a single-element group and a two-element semilattice from our further considerations.

In other words, Φ is transitive when $\bigcup \Phi = A \times A$, where $\bigcup \Phi$ denotes the set-theoretical union of all elements of Φ (recall that each element of Φ is a special subset of $A \times A$).

Observe that *permutations* of A (that is, one-to-one mappings of A onto itself) form an inverse subsemigroup $\mathcal{G}_A$ of $\mathcal{I}_A$. Clearly, $\mathcal{G}_A$ is the symmetric group of permutations of A.

[1] We denote the empty partial transformation by the same symbol as the empty set because, for us, every partial transformation φ *is* a set. Namely, $\varphi = \{(a, b) \in A \times A \mid a\varphi = b\}$. From this point of view the empty partial transformation *is* the empty set.

[2] What is so noble in noble inverse semigroups? We need a term for them, and the word "noble" has already been used in [9].