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The relation type of affine algebras and algebraic varieties

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ABSTRACT

We introduce the notion of relation type of an affine algebra and prove that it is well defined by using the Jacobi–Zariski exact sequence of André–Quillen homology. In particular, the relation type is an invariant of an affine algebraic variety. Also as a consequence of the invariance, we show that in order to calculate the relation type of an ideal in a polynomial ring one can reduce the problem to trinomial ideals. When the relation type is at least two, the extreme equidimensional components play no role. This leads to the non-existence of affine algebras of embedding dimension three and relation type two.

1. Introduction

Let $A = R/I = k[x_1, \ldots, x_n]/I$ be an affine k-algebra, where k is a field, $R = k[x_1, \ldots, x_n]$ is the polynomial ring in the variables $x_1, \ldots, x_n$ over k and $I = (f_1, \ldots, f_s)$ is an ideal of R.

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In this note we introduce the following invariant of A: the *relation type of* A is defined as $\mathrm{rt}(A) = \mathrm{rt}(I)$, where $\mathrm{rt}(I)$ stands for the relation type of the ideal I.

Recall that if $\mathbf{R}(I) = R[It] = \oplus_{q\geq 0} I^q t^q$ is the *Rees algebra* of I and $\varphi : S = R[t_1, \ldots, t_s] \to \mathbf{R}(I)$ is the natural graded polynomial presentation sending t_i to $f_i t$, then $L = \ker(\varphi) = \oplus_{q\geq 1} L_q$, referred to as the ideal of *equations* of I, is a graded ideal of S and the relation type of I, denoted by $\mathrm{rt}(I)$, is the least integer $N \geq 1$ such that L is generated by its components of degree at most N. Concerning the equations of an ideal and its relation type, see for instance, and with no pretense of being exhaustive, [6–8,13–15,17–19,21,22], and the references therein.

By means of the Jacobi–Zariski exact sequence of André–Quillen homology, we prove that the definition of $\mathrm{rt}(A)$ does not depend on the presentation of A. In particular, we obtain an invariant for affine algebraic varieties. Concretely, if V is an affine algebraic k-variety, the *relation type* of V is defined as $\mathrm{rt}(V) = \mathrm{rt}(k[V])$, the relation type of its coordinate ring $k[V]$. Another consequence is that in order to calculate the relation type of an ideal in a polynomial ring one can reduce the problem to ideals generated by trinomials, though at the cost of introducing more generators and more variables.

We then study the connection between the equidimensional decomposition of a radical ideal I of $R = k[x_1, \ldots, x_n]$ and its relation type $\mathrm{rt}(I)$. We conclude that the equidimensional components of dimension 1 and n are not relevant whenever the relation type is at least two. As a corollary we obtain a somewhat surprising result, namely, that there are no affine k-algebras of embedding dimension three and relation type two.

Due to the aforementioned result, the examples we provide are essentially focussed on affine space curves. At this point one should emphasize that the explicit calculation of the equations of an ideal is computationally a very expensive task. The reduction to trinomial ideals, unfortunately, does not seem to improve, in general, the approach to the problem. It would be desirable to obtain a wide range of irreducible affine space curves with prescribed relation type. It would also be interesting to understand better the geometric meaning of the relation type of an algebraic variety.

Notice that, to our knowledge, there is at least another notion also named *relation type of an algebra*. Indeed, in [20, Definition 2.7], W.V. Vasconcelos defines the relation type of a standard algebra $A = k[x_1, \ldots, x_n]/I$ as the least integer s such that $I = (I_1, \ldots, I_s)$, where $I \subset (x_1, \ldots, x_n)^2$ is a homogeneous ideal of the polynomial ring $k[x_1, \ldots, x_n]$ over the field k and I_i is the i-th graded component of I. It is clear that both definitions do not coincide, even in the homogeneous case; for instance, take $I = (f)$ a principal ideal of $k[x_1, \ldots, x_n]$ generated by a homogeneous polynomial f of degree $p \geq 2$.

The paper is organized as follows. In Section 2, we set the notations used throughout. In Section 3, we prove the invariance theorem and the reduction to trinomials. Section 4 is devoted to study the effect of an equidimensional decomposition in the computation of the relation type. Finally, in Sections 5 and 6, we give some examples in embedding dimension three.