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Derived categories of surfaces isogenous to a higher product

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ABSTRACT

Let $S = (C \times D)/G$ be a surface isogenous to a higher product of unmixed type with $p_g = q = 0$, $G = (\mathbb{Z}/3)^2$. We construct exceptional sequences of line bundles of maximal length on S. As a consequence we find new examples of quasiphantom categories.

1. Introduction

Recently derived categories of surfaces of general type attracted a lot of attention. Several interesting semiorthogonal decompositions of the derived categories were constructed by Böhning, Graf von Bothmer, and Sosna on the classical Godeaux surface [7]; Alexeev and Orlov on the primary Burniat surfaces [1]; Galkin and Shinder on the Beauville surface [12]; Böhning, Graf von Bothmer, Katzarkov and Sosna on the determinantal Barlow surfaces [6]; Fakhruddin on some fake projective planes [11]; Galkin, Katzarkov, Mellit and Shinder on some different fake projective planes and on a fake cubic surface [13]; Coughlan on some surfaces obtained as abelian coverings of del Pezzo surfaces [8]; Keum

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on some fake projective planes with enough automorphisms [16]. These semiorthogonal decompositions consist of admissible subcategories generated by exceptional sequences of line bundles of maximal lengths and their orthogonal complements. These orthogonal complements have vanishing Hochschild homology groups. An admissible triangulated subcategory of a derived category of a smooth projective variety is called a quasiphantom category if its Hochschild homology group vanishes and its Grothendieck group is finite. When the Grothendieck group of a quasiphantom category also vanishes it is called a phantom category. Gorchinskiy and Orlov in [14] constructed phantom categories using quasiphantom categories constructed in [1,7,12]. Determinantal Barlow surfaces also provide examples of phantom categories [6].

Let S be a surface isogenous to a higher product $(C \times D)/G$ of unmixed type with $p_g = q = 0$. If G is an abelian group, Bauer and Catanese [2] proved that G is one of $(\mathbb{Z}/2)^3, (\mathbb{Z}/2)^4, (\mathbb{Z}/3)^2, (\mathbb{Z}/5)^2$. As mentioned above, Galkin and Shinder [12] constructed exceptional sequences of line bundles of maximal length 4 on the Beauville surface which is a surface isogenous to a higher product with $p_g = q = 0$ and $G = (\mathbb{Z}/5)^2$. Motivated by their work, we study the derived categories of the 2-dimensional family of surfaces isogenous to a higher product with $p_g = q = 0$, $G = (\mathbb{Z}/3)^2$ and prove that there exist similar semiorthogonal decompositions.

Theorem 1.1. *Let $S = (C \times D)/G$ be a surface isogenous to a higher product with $p_g = q = 0$, $G = (\mathbb{Z}/3)^2$. There are exceptional sequences of line bundles of maximal length 4 in $D^b(S)$ and the orthogonal complements of the admissible subcategories generated by these line bundles are quasiphantom categories.*

This result gives new examples of quasiphantom categories having Grothendieck groups $(\mathbb{Z}/3)^5$ and these categories can be used to construct phantom categories by a theorem of Gorchinskiy and Orlov [14]. We also compute Hochschild cohomology groups of quasiphantom categories and prove that for some exceptional sequences we obtained the categories generated by those exceptional sequences are deformation invariant. While adding these results to this paper which was on the arXiv, similar results have been obtained independently by Coughlan in [8] via different method. In his paper [8], Coughlan considers general type surfaces which are obtained as abelian covers of del Pezzo surfaces satisfying some conditions. His method can be applied to surfaces isogenous to a higher product with $G = (\mathbb{Z}/3)^2$, $G = (\mathbb{Z}/5)^2$ and many other surfaces of general type. He constructs many exceptional sequences of maximal lengths on these surfaces and studies deformation invariance and Hochschild cohomology groups.

This paper is organized as follows. In Section 2, we collect some basic facts about the surfaces isogenous to a higher product and compute the Grothendieck groups of these surfaces. In Section 3, we construct exceptional sequences of line bundles on the 2-dimensional family of surfaces isogenous to a higher product with $p_g = q = 0$, $G = (\mathbb{Z}/3)^2$. In Section 4, we discuss quasiphantom and phantom categories. In Section 5, we will consider the cases where $G = (\mathbb{Z}/2)^3$, $G = (\mathbb{Z}/2)^4$.