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Recovering the Lie algebra from its extremal geometry

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ABSTRACT

An element x of a Lie algebra L over the field F is extremal if $[x, [x, L]] = Fx$. Under minor assumptions, it is known that, for a simple Lie algebra L, the extremal geometry $\mathcal{E}(L)$ is a subspace of the projective geometry of L and either has no lines or is the root shadow space of an irreducible spherical building Δ. We prove that if Δ is of simply-laced type, then L is a quotient of a Chevalley algebra of the same type.

1. Introduction

Lie theory is an active and important field whose applications are widespread throughout mathematics and physics. Underlying objects in Lie theory are the simple Lie algebras which have striking connections to algebraic groups and buildings. Simple complex Lie algebras were classified by Killing [8] and Cartan [1] in the late nineteenth century. In the second half of the twentieth century, through the efforts of many mathematicians, a classification of all simple finite-dimensional Lie algebras over algebraically closed fields with characteristic at least 5 was achieved (cf. [12,13]). The two main classes of Lie algebras that emerge from the result are the classical Lie algebras derived from the simple complex Lie algebras and the Lie algebras of Cartan type derived from the Witt algebras.

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Given a simple complex Lie algebra $\mathfrak{g}$ corresponding to a root system Φ of type X_n, we can construct its $\mathbb{Z}$-form $\mathfrak{g}_{\mathbb{Z}}$ by taking the integral span of a Chevalley basis of $\mathfrak{g}$. By tensoring $\mathfrak{g}_{\mathbb{Z}}$ with an arbitrary field F we obtain the Chevalley algebra $\mathfrak{g}_F$. A *classical Lie algebra* (or simply a *classical algebra*) of type X_n over F is defined as an arbitrary nonzero quotient of the Chevalley algebra $\mathfrak{g}_F$. For classical Lie algebras we have the familiar notions such as Cartan subalgebra, root elements, etc.

Extremal elements are extensively utilised in the classification of simple modular Lie algebras over algebraically closed fields and there is good evidence to believe that they should play a crucial role for arbitrary fields. An element x of a Lie algebra L over a field F is *extremal* if $[x,[x,L]] \subseteq Fx$ and such an element x is a *sandwich* if $[x,[x,L]] = 0$ (additional conditions are needed when F has characteristic 2). Examples of extremal elements are the long root elements of the classical algebras. In [10], Premet shows that any simple Lie algebra over an algebraically closed field of characteristic at least 5 contains a nonzero extremal element. It is shown in [5] that, with one exception, a simple Lie algebra over an arbitrary field of characteristic at least 5 that contains a non-sandwich extremal element is generated by such elements.

Extremal elements, first formally introduced in [6], are also the focal point of the effort spearheaded by Cohen et al. to provide a geometric characterisation of the classical Lie algebras. In [3], Cohen and Ivanyos define a point-line space on the set of nonzero extremal elements of L called the *extremal geometry* denoted by $\mathcal{E}(L)$. By combining the results in [3,4,7], one can conclude that if L is finite-dimensional, simple and generated by extremal elements with no sandwiches, then $\mathcal{E}(L)$ either contains no lines or is the root shadow space of a spherical building. An interesting and important question is whether one can recover the algebra L from the geometry $\mathcal{E}(L)$. By using the theory of buildings first introduced by Tits in [14], this would provide a geometric interpretation of the classical Lie algebras analogous to his own achievement for the algebraic groups. In the PhD thesis of the second author [11] this question was addressed in the case of simply-laced types and, in particular, a complete solution was obtained for the diagram A_n. In this paper we provide a uniform treatment of all simply-laced diagrams. Namely, we show that if $\mathcal{E}(L)$ is of simply-laced type, then L is indeed a classical Lie algebra of the same type.

Theorem 1.1. *Suppose that L is a finite-dimensional simple Lie algebra generated by its extremal elements and contains no sandwiches. Furthermore, suppose that the extremal geometry $\mathcal{E}(L)$ is the root shadow space of a building of type A_n ($n \geq 2$), D_n ($n \geq 4$), E_6, E_7 or E_8. Then L is a classical Lie algebra of the same type as $\mathcal{E}(L)$.*

The paper is organised in the following way. In Section 2, we describe the relationship between the root shadow space of an apartment, wherein points are described as arctic regions, and the corresponding root system. We survey the known results concerning root shadow spaces and construct a dictionary between root shadow spaces and two different interpretations of root systems. In Section 3, we introduce root filtration spaces that are