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On the highest multiplicity locus of algebraic varieties and Rees algebras

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Let X be an equidimensional scheme of finite type over a perfect field k. Under these conditions, the multiplicity along points of X defines an upper semi-continuous function, say $\text{mult}_X : X \to \mathbb{N}$, which stratifies X into its locally closed level sets. We study this stratification, and the behavior of the multiplicity when blowing up at regular equimultiple centers. We also discuss a natural compatibility of these two concepts when X is replaced with its underlying reduced scheme. The main result in this paper is to show that, given a variety X, there is a well defined Rees algebra over X, naturally attached to maximum value of the multiplicity.

1. Introduction

Let X be an equidimensional scheme of finite type over a perfect field k. The goal is to attach a Rees algebra to X, and more precisely, to the closed set of points of maximum

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multiplicity in X. The Rees algebra will be trivial when X is smooth, and in general, it will appear as an algebraic structure attached to set of points of highest multiplicity in X. Rees algebras have been studied thoroughly in commutative algebra due to their various applications (see e.g. [15]).

The multiplicity function, $\mathrm{mult}_X : X \to \mathbb{N}$, is defined by setting $\mathrm{mult}_X(x)$ as the multiplicity of the local ring $\mathcal{O}_{X,x}$ for each $x \in X$. A theorem of Dade [5] states that this is an upper semi-continuous function, and therefore its level sets define a stratification of X in locally closed sets.

Another way to approach this upper semi-continuity is by using results of Bennett [1] concerning the Hilbert–Samuel function on excellent schemes. In fact, the multiplicity of a local ring can be expressed in terms of the Hilbert polynomial. Bennett proves that the Hilbert–Samuel function is upper semi-continuous on X, and it is not hard to infer from this that mult_X is upper semi-continuous.

Denote by X_{red} the reduced scheme associated to X. Here we may identify X with its underlying topological space, which coincides with that of X_{red}. As X_{red} is also equidimensional, $\mathrm{mult}_{X_{\mathrm{red}}} : X \to \mathbb{N}$ is upper semi-continuous. Throughout Sections 2 and 3, we discuss some natural properties concerning the stratification defined by the multiplicity on both X and X_{red}. First, we show that the behavior of the multiplicity on X can be easily reduced to that of its irreducible components (Proposition 2.8). Moreover, we show that the stratifications of X and X_{red} defined by the multiplicity are essentially the same (Lemma 3.2).

We will also draw our attention to the behavior of the multiplicity when blowing up along regular and equimultiple centers. For us, these will be called *permissible* blow-ups as opposed to Hironaka, for whom a permissible center is a regular one contained in the maximum Hilbert–Samuel stratum. Let $X \xleftarrow{\pi} X_1$ denote a permissible blow-up. Dade proved that $\mathrm{mult}_{X_1}(x_1) \leq \mathrm{mult}_X(\pi(x_1))$ for any $x_1 \in X_1$ (see [5]). This result was later generalized and simplified by Orbanz [13]. In particular, it implies that $\max \mathrm{mult}_{X_1} \leq \max \mathrm{mult}_X$.

Consider an equidimensional scheme X. If the multiplicity is constant along X, then X_{red} must be regular. Otherwise, Dade's result leads naturally to the following question: if the multiplicity is not constant along X, does there exist a sequence of permissible blow-ups, say

$$X \longleftarrow X_1 \longleftarrow \cdots \longleftarrow X_n,$$

so that $\max \mathrm{mult}_{X_n} < \max \mathrm{mult}_X$? Hironaka posed this problem in the case that X is a variety as Question (D) in [7] (p. 134). The problem has already been solved for schemes defined over a field of characteristic zero (Theorem 8.11 [17]). In Section 3, we show that, whenever such a sequence exists, it is naturally compatible with taking reduction (see Proposition 3.7). This compatibility was already pointed out in Remark 6.10 [17]. Here, we analyze it from a different perspective.