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Inflectional loci of quadric fibrations

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ABSTRACT

Quadric fibrations over smooth curves are investigated with respect to their osculatory behavior. In particular, bounds for the dimensions of the osculating spaces are determined, and explicit formulas for the classes of the inflectional loci are exhibited under appropriate assumptions. Moreover, a precise description of the inflectional loci is provided in several cases. The associated projective bundle and its image in the ambient projective space of the quadric fibration, the enveloping ruled variety, play a significant role. Several examples are discussed to illustrate concretely the various situations arising in the analysis.

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1. Introduction

The osculatory behavior of scrolls in $\mathbb{P}^N$ has been investigated in many papers. The fact that they contain many lines implies that the dimension of every k-th osculating space is considerably smaller than what is expected for a general smooth projective variety. In particular, for n-dimensional scrolls over curves this dimension does not exceed kn and assuming that equality holds at a general point one can describe the k-th inflectional locus and its cohomology class [17]. Recently, an analogous investigation has been carried out also for scrolls over projective varieties of dimension ≥ 2 [18].

A natural question arising from these studies (which we have also been asked when presenting our results) is: what about other special varieties, in particular quadric fibrations over a curve? The aim of this paper is to approach this question. Actually, we focus on the case $k = 2$ and we analyze the relationships between the osculatory behavior of such varieties and several other aspects of their geometry, like linear normality, embedding in a scroll, ampleness, etc.

The case of quadric fibrations looks particularly nice since any quadric fibration $X \subset \mathbb{P}^N$ over a smooth curve C is naturally contained as a divisor inside a projective bundle P over C, and the embedding of X in $\mathbb{P}^N$ extends to a morphism $\varphi : P \to \mathbb{P}^N$ to the same projective space, which maps every linear fiber of P isomorphically to the linear span of the corresponding fiber of X in $\mathbb{P}^N$. This map φ, however, is not always an embedding, which turns out to be equivalent to the fact that its image $R = \varphi(P)$ may not be a scroll over C.

In a sense this contrasts with the naive expectation that the inflectional locus of X should be determined by that of R. On the one hand, when φ is an embedding, looking at the pair (P, X) one can compare the osculatory behavior of X with respect to that of R along X. In particular, letting Φ_2 denote the second inflectional locus, we have that $\Phi_2(X) \supseteq X \cap \Phi_2(R)$ and we have examples showing that this is not an equality in general. On the other hand, $\Phi_2(X)$ always contains the set S of singular points of singular fibers of X if $n \geq 3$ and all singular fibers if $n = 2$, facts which are not evident if we simply look at the pair (X, P), because the linear span of a fiber F of X is a linear $\mathbb{P}^n$ inside R regardless of whether F is a smooth or a singular fiber.

We realize the role of S via another, more direct geometric approach, looking at $X \subset \mathbb{P}^N$ by itself, and at the linear subsystem of hyperplane sections of X having a given point $x \in X$ as a singular point of multiplicity 3 (Theorem 11).

We want to emphasize that everywhere we work without the assumption that X is linearly normally embedded. This allows us to put in evidence flexes arising from isomorphic projections or even hypo-osculation phenomena deriving from them (Section 7).

Among the results, we mention the upper bound we obtain for the highest dimension σ_k of a k-th osculating space to an n-dimensional quadric fibration X: if N is large enough we have $\sigma_k \leq k(n+1) - 1$ (Corollary 14). Observe that for $k = n = 2$ this is the same natural bound occurring for any smooth surface, while this is not the case for $k = 2$ in higher dimension and for $k \geq 3$ even for $n = 2$. As a consequence, conic fibrations have