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Differential forms and bilinear forms under field extensions

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ABSTRACT

Let F be a field of characteristic $p > 0$. Let $\Omega^n(F)$ be the F-vector space of n-differentials of F over F^p. Let $K = F(g)$ be the function field of an irreducible polynomial g in $m \geqslant 1$ variables over F. We derive an explicit description of the kernel of the restriction map $\Omega^n(F) \to \Omega^n(K)$. As an application in the case $p = 2$, we determine the kernel of the restriction map when passing from the Witt ring (resp. graded Witt ring) of symmetric bilinear forms over F to that over such a function field extension K.

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1. Introduction

When studying algebraic objects defined over some base field, such as quadratic forms, central simple algebras or Milnor K-groups, it is quite natural to ask how these objects behave when extending scalars to a field extension. In particular, one would like to be able to determine the kernel of the restriction homomorphism between the Witt rings, the Brauer groups, and the Milnor K-groups of the base field and of the extension field.

One of our main objectives is to derive an explicit description of the Witt kernel for symmetric bilinear forms in characteristic 2 for a large class of field extensions, namely extensions given by function fields of arbitrary hypersurfaces over the base field. This includes all finite simple extensions of the base field (the one-variable case). Our proof is based on a study of how the space of differential forms $\Omega^m(F)$ behaves under field extensions. These results on differential forms are of interest in their own right and they are proved for fields of arbitrary positive characteristic $p > 0$.

The strategy of the proof for determining the Witt kernels is as follows. The crucial ingredient is the determination of the kernel $\Omega^m(E/F)$ of the map $\Omega^m(F) \to \Omega^m(E)$ where E is the quotient field of $F[X]/(f(X))$ for an irreducible polynomial $f(X) \in F[X]$ where $X = (X_1, \ldots, X_n)$, and $\mathrm{char}(F) = p > 0$. This is done in several steps. We start by recalling that $\Omega^m(E/F) = 0$ if E/F is purely transcendental or separable algebraic. This allows us to discard all irreducible polynomials that are not in $F[X^p] = F[X_1^p, \ldots, X_n^p]$. Next, we treat the case $n = 1$ (i.e. the case of a simple algebraic extension). The kernel can then be expressed as the subspace of elements in $\Omega^m(F)$ that are annihilated by all differentials da where $a \in F^*$ runs through all nonzero coefficients of $f(X)$ (which we may assume to be monic). This is done in Section 7. We then use an induction on the number of variables n. If $n \geqslant 2$, let $X' = (X_1, \ldots, X_{n-1})$. By a linear change of variables and suitable scaling, we then may assume that $f(X) \in F[X^p]$ is monic in X_n with coefficients in $F[X']$, so we use the 1-variable case to conclude that over $F(X')$, the kernel consists of those elements in $\Omega^n(F)$ that, over $F(X')$, are annihilated by differentials $da(X')$ where $a(X') \in F[X'^p]$ runs through the coefficients of $f(X) \in F(X')[X_n]$. By an induction on the number of variables, we can then show that the elements in $\Omega^m(F)$ annihilated by $da(X')$ are already annihilated by the $d\alpha$ where $\alpha \in F^*$ runs through all nonzero F-coefficients of $a(X')$. This induction is done in Section 8 where we also need some results on the kernel $\Omega^m(E/F)$ in the special case where E is the function field of a quasilinear p-form. The main result of this section (Theorem 8.5) states that for irreducible polynomials $f \in F[X^p]$, the kernel $\Omega^m(E/F)$ is the subspace annihilated by all $d(a/b)$ where a and b are nonzero values represented by f over F, or, equivalently, by all $d(a/b)$ where a and b are nonzero coefficients of f.

This description of the kernel $\Omega^m(E/F)$ has been used by S. Scully [30] to determine the kernel of the map $K_m^M(F)/pK_m^M(F) \to K_m^M(E)/pK_m^M(E)$ of Milnor K-theory modulo p for a field F of characteristic p and $E = F(f)$ a function field over F in the above