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Gradings on the Lie algebra D_4 revisited

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ABSTRACT

We classify group gradings on the simple Lie algebra $\mathcal{L}$ of type D_4 over an algebraically closed field of characteristic different from 2: fine gradings up to equivalence and G-gradings, with a fixed group G, up to isomorphism. For each G-grading on $\mathcal{L}$, we also study graded $\mathcal{L}$-modules (assuming characteristic 0).

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1. Introduction

In the past two decades, there has been much interest in gradings on simple Lie algebras by arbitrary groups — see our recent monograph [8] and the references therein. In particular, the classification of fine gradings (up to equivalence) on all finite-dimensional simple Lie algebras over an algebraically closed field of characteristic 0 is essentially complete (see [8, Chapters 3–6], [6,20]). For a given group G, the classification of G-gradings (up to isomorphism) on classical simple Lie algebras over an algebraically closed field of characteristic different from 2 was done in [1] (see also [8, Chapter 3]), excluding type D_4, which exhibits exceptional behavior due to the phenomenon of triality. Although the case of D_4 is included in [5] (see also [3] and [8, §6.1]), only fine gradings are treated there and the characteristic is assumed to be 0. Since we do not see how to extend those arguments to positive characteristic, here we use an approach based on affine group schemes, which was also employed in [1].

Let $\mathbb{F}$ be the ground field. Except in the Preliminaries, we will assume $\mathbb{F}$ *algebraically closed* and $\operatorname{char}\mathbb{F} \neq 2$. All vector spaces, algebras, tensor products, group schemes, etc., will be assumed over $\mathbb{F}$ unless indicated otherwise. The superscript $\times$ will indicate the multiplicative group of invertible elements.

Recall that affine group schemes are representable functors from the category $\mathrm{Alg}_{\mathbb{F}}$ of unital associative commutative algebras over $\mathbb{F}$ to the category of groups — we refer the reader to [18], [13, Chapter VI] or [8, Appendix A] for the background. Every (naïve) algebraic group gives rise to an affine group scheme. These are precisely the *smooth algebraic* group schemes, i.e., those whose representing (Hopf) algebra is finitely generated and reduced. In characteristic 0, all group schemes are reduced, but it is not so in positive characteristic. We will follow the common convention of denoting the (smooth) group schemes corresponding to classical groups by the same letters, but using bold font to distinguish the scheme from the group (which is identified with the $\mathbb{F}$-points of the scheme). It is important to note that this convention should be used with care: for example, the automorphism group scheme $\mathbf{Aut}_{\mathbb{F}}(\mathcal{U})$ of a finite-dimensional algebra $\mathcal{U}$ is defined by $\mathbf{Aut}_{\mathbb{F}}(\mathcal{U})(\mathcal{S}) = \mathrm{Aut}_{\mathcal{S}}(\mathcal{U}\otimes\mathcal{S})$ for every $\mathcal{S}$ in $\mathrm{Alg}_{\mathbb{F}}$, and may be strictly larger than the smooth group scheme corresponding to the algebraic group $\mathrm{Aut}_{\mathbb{F}}(\mathcal{U})$.

Let $\mathcal{L}$ be a Lie algebra of type D_4. It is well known that the automorphism group scheme $\mathbf{Aut}_{\mathbb{F}}(\mathcal{L})$ is smooth and we have a short exact sequence

$$\mathbf{1} \longrightarrow \mathbf{PGO}_8^+ \longrightarrow \mathbf{Aut}_{\mathbb{F}}(\mathcal{L}) \xrightarrow{\pi} \mathbf{S}_3 \longrightarrow \mathbf{1} \tag{1}$$

where $\mathbf{PGO}_8^+$ is the group scheme of inner automorphisms (which corresponds to the algebraic group PGO_8^+, see e.g. [13, §12.A]) and $\mathbf{S}_3$ is the constant group scheme corresponding to the symmetric group S_3. This sequence can be split by identifying $\mathcal{L}$ with the *triality Lie algebra* (see Definition 5 below) of the *para-Cayley algebra* $\mathcal{C}$, i.e., the Cayley algebra equipped with the new product $x \bullet y = \bar{x}\bar{y}$, where juxtaposition denotes the usual product of $\mathcal{C}$ and bar denotes its standard involution (see e.g. [8, §4.1]). If