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Factorization theory: From commutative to noncommutative settings ☆

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A B S T R A C T

We study the non-uniqueness of factorizations of non zero-divisors into atoms (irreducibles) in noncommutative rings. To do so, we extend concepts from the commutative theory of non-unique factorizations to a noncommutative setting. Several notions of *factorizations* as well as *distances* between them are introduced. In addition, arithmetical invariants characterizing the non-uniqueness of factorizations such as the *catenary degree*, the ω*-invariant*, and the *tame degree*, are extended from commutative to noncommutative settings. We introduce the concept of a cancellative semigroup being *permutably factorial*, and characterize this property by means of corresponding catenary and tame degrees. Also, we give necessary and sufficient conditions for there to be a weak transfer homomorphism from a cancellative semigroup to its reduced abelianization. Applying the abstract machinery we develop, we determine various catenary degrees for classical maximal orders in central simple algebras over global fields by using a natural transfer homomorphism to a monoid of zero-sum sequences over a ray class group. We also determine catenary degrees and the permutable tame degree for the semigroup of non zero-divisors of the ring of $n \times n$ upper

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triangular matrices over a commutative domain using a weak transfer homomorphism to a commutative semigroup.

1. Introduction

The study of factorizations in commutative rings and semigroups has a long and rich history. Beginning with attempts to understand the factorizations of elements in rings of algebraic integers into irreducibles, this field has grown to include the investigation of non-unique factorizations in Mori domains, Krull domains and Krull monoids, including the study of direct-sum decompositions of modules (see [5]). These investigations have used tools from multiplicative ideal theory, algebraic and analytic number theory, combinatorics, and additive group theory. A thorough overview of the various aspects of commutative factorization theory can be found in [3,9,13,24,25,30].

On the other hand, the study of unique and non-unique factorization in noncommutative rings and semigroups has received limited attention. In fact, for many years the study of factorizations in noncommutative settings had been restricted to characterizing and studying noncommutative rings with properties analogous to that of commutative unique factorization domains or to studying factorizations of certain (symmetric) polynomials over noncommutative (e.g. matrix) rings (see [32,35,41,44,31,27,42,21,50,40]). From the beginning it was clear that each (noncommutative) PID intrinsically has certain unique factorization properties (see, for instance, [36, Chapter 3.4], [20, Chapter VI.9] and [48, page 230]). More recently, such phenomena have been studied; for semifirs and in particular 2-firs by P.M. Cohn [16,17], for the ring of Hurwitz and Lipschitz quaternions by Conway and Smith [18] and by H. Cohn and Kumar [15], for quaternion orders by Estes and Nipp [22,23], and in a more general setting by Brungs [8]. Somewhat different notions of unique factorization domains and unique factorization rings were introduced by Chatters and Jordan [12,14,37], and have found applications in [38,43,34].

Recently, techniques from the factorization theory of commutative rings and monoids have been used to investigate non-unique factorizations in a noncommutative setting. For example, in [7] factorizations within some natural subsemigroups of matrices with integer coefficients are considered, and in [4] factorizations within the subsemigroup of non zero-divisors of the ring of $n \times n$ upper triangular matrices $T_n(D)^\bullet$ over an arbitrary atomic commutative domain D are studied. In [26], noncommutative Krull monoids are investigated. Through the study of the divisorial two-sided ideals of S that closely parallels the techniques that have been used fruitfully for commutative Krull monoids, it is shown that in the normalizing case ($aS = Sa$ holds for all a in the Krull monoid S) many results from the commutative setting generalize. In [51] this approach is, by means of divisorial one-sided ideal theory, extended to a class of semigroups that includes commutative and normalizing Krull monoids as special cases. In particular, this is applied