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The weighted universal Coxeter group and some related conjectures of Lusztig ☆

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ABSTRACT

Lusztig proposed 15 conjectures for any weighted Coxeter groups. In this study, we explicitly describe Kazhdan–Lusztig cells, distinguished involutions and the function **a** for any weighted universal Coxeter group. We also verify these conjectures for this group.

0. Introduction

Let W be a Coxeter group and S is its Coxeter generator set. Let $\mathcal{H}(W)$ be the associated Iwahori–Hecke algebra of W (see [6]). In order to construct representations of W and $\mathcal{H}(W)$, Kazhdan and Lusztig introduced certain equivalence classes, called left cells, right cells and two-sided cells, of W in [6]. Later, in [9], Lusztig defined a weight function L on (W, S), a weighted Coxeter group (W, S, L) and the associated Iwahori–Hecke algebra $\mathcal{H}(W, S, L)$ (see 1.1), as well as defining the cells of (W, S, L), and he proposed 15 conjectures for extending some results on cells to the weighted case.

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It is useful to describe all of the cells explicitly for any given weighted Coxeter group (W, S, L) as L ranges over all the possible weight functions on W, particularly for those that are not constant on S. This was achieved when W is of types I_m (m is either even or ∞), F_4, $\widetilde{C}_2$ and $\widetilde{G}_2$ (see [9,3,4]) and partially when W is of types B_n, $\widetilde{C}_n$ and $\widetilde{B}_n$ for $n > 2$ (see [8,1,5,12,13,10]). When (W, S) is either a finite or affine Weyl group, the 15 conjectures of Lusztig were verified if L is either the length function or (W, S, L) is in the quasi-split case (in the sense of Lusztig, see [9, Chapters 15–16]). In the present study, we consider the case where (W, S, L) is any weighted universal Coxeter group. First, we study the properties of some structural coefficients of the associated Iwahori–Hecke algebra, before describing left cells, two-sided cells, the function $\mathbf{a}$ and distinguished involutions explicitly for (W, S, L) (Theorems 4.3, 4.4 and 3.7, and Lemma 5.3 (2)). We conclude that any left (respectively, two-sided) cell of (W, S, L) is left-connected (respectively, two-sided connected) (Remark 4.5 (2)). We then verify the 15 conjectures of Lusztig for (W, S, L) (Theorem 5.2).

The remainder of this paper is organized as follows. We provide some concepts, notations and known results in Section 1. In the following sections, we consider the weighted universal Coxeter group (W, S, L). In Sections 2–3, we study the structural coefficients $f_{x,y,z}$, $\mu^s_{y,w}$ and $\gamma_{x,y,z}$ of $\mathcal{H}(W, S, L)$ and the function $\mathbf{a}$ on W. In Section 4, we describe all the cells for (W, S, L). Finally, we verify the 15 conjectures of Lusztig for (W, S, L) in Sections 5–6.

1. Preliminaries

In the present section, we provide some concepts, notations and known results for later use. Most of them follow from those given by Lusztig in [9].

1.1. Let $\mathbb{Z}$ (respectively, $\mathbb{N}$, and $\mathbb{P}$) denote the set of integers (respectively, non-negative integers, and positive integers). For any $i \leqslant j$ in $\mathbb{Z}$, denote $[i, j]$ as the set $\{i, i+1, \ldots, j\}$ and denote $[1, j]$ simply by $[j]$.

Let W be a Coxeter group and S is its Coxeter generator set. Let ℓ be the length function and $\leqslant$ is the Bruhat–Chevalley order on (W, S). Denote $L : W \to \mathbb{Z}$ as a *weight function* on W if $L(xy) = L(x) + L(y)$ for any $x, y \in W$ with $\ell(xy) = \ell(x) + \ell(y)$. Hence $L(s) = L(t)$ for any $s, t \in S$ conjugate in W. Denote (W, S, L) as a *weighted Coxeter group*.

Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ be the ring of Laurent polynomials in an indeterminate v with integer coefficients. Denote $v_w := v^{L(w)}$ for $w \in W$. By definition, the Iwahori–Hecke algebra $\mathcal{H} := \mathcal{H}(W, S, L)$ of (W, S, L) is the associative $\mathcal{A}$-algebra with an $\mathcal{A}$-basis $\{T_w \mid w \in W\}$, subject to the multiplication rule:

$$\begin{aligned} T_s^2 &= (v_s - v_s^{-1})T_s + T_e && \text{for } s \in S, \\ T_xT_y &= T_{xy} && \text{for } x, y \in W \text{ with } \ell(xy) = \ell(x) + \ell(y), \end{aligned} \tag{1.1.1}$$