Journal of Algebra 439 (2015) 1–32

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

The largest strong left quotient ring of a ring

V.V. Bavula

Department of Pure Mathematics, University of Sheffield, Hicks Building, Sheffield S3 7RH, UK

A R T I C L E   I N F O

Article history:
Received 9 September 2014
Available online 26 May 2015
Communicated by Louis Rowen

MSC:
16S85
16U20
16P50
16P60
16P20

Keywords:
The (largest) strong left quotient ring of a ring
Goldie's Theorem
The strong left localization radical
The left quotient ring of a ring
The largest left quotient ring of a ring
A maximal left denominator set
The left localization radical of a ring

A B S T R A C T

For an arbitrary ring R, the *largest strong left quotient ring* $Q_l^s(R)$ of R and the *strong left localization radical* $\mathfrak{l}_R^s$ are introduced and their properties are studied in detail. In particular, it is proved that $Q_l^s(Q_l^s(R)) \simeq Q_l^s(R)$, $\mathfrak{l}^s_{R/\mathfrak{l}_R^s} = 0$ and a criterion is given for the ring $Q_l^s(R)$ to be a semisimple ring. There is a canonical homomorphism from the classical left quotient ring $Q_{l,cl}(R)$ to $Q_l^s(R)$ which is not an isomorphism, in general. The objects $Q_l^s(R)$ and $\mathfrak{l}_R^s$ are explicitly described for several large classes of rings (semiprime left Goldie ring, left Artinian rings, rings with left Artinian left quotient ring, etc.).

Contents

E-mail address: v.bavula@sheffield.ac.uk.

http://dx.doi.org/10.1016/j.jalgebra.2015.04.037
0021-8693/

1. Introduction

The aim of the paper is, for an arbitrary ring R, to introduce new concepts: *the largest strong left denominator set* $T_l(R)$ *of* R*, the largest strong left quotient ring* $Q_l^s(R) := T_l(R)^{-1}R$ *of* R *and the strong left localization radical* $\mathfrak{l}_R^s$ *of* R, and to study their properties.

In this paper, the following notation is fixed:

- R is a ring with 1 and R^* is its group of units;
- $\mathcal{C} = \mathcal{C}_R$ is the set of *regular* elements of the ring R (i.e. $\mathcal{C}$ is the set of non-zero-divisors of the ring R);
- ${}'\mathcal{C}_R$ is the set of *left regular* elements of the ring R, i.e. ${}'\mathcal{C}_R := \{c \in R \mid \ker(\cdot c) = 0\}$ where $\cdot c : R \to R$, $r \mapsto rc$;
- $Q = Q_{l,cl}(R) := \mathcal{C}^{-1}R$ is the *left quotient ring* (the *classical left ring of fractions*) of the ring R (if it exists, i.e. if $\mathcal{C}$ is a left Ore set) and Q^* is the group of units of Q;
- $\mathrm{Ore}_l(R) := \{S \mid S \text{ is a left Ore set in } R\}$;
- $\mathrm{Den}_l(R) := \{S \mid S \text{ is a left denominator set in } R\}$;
- $\mathrm{Ass}_l(R) := \{\mathrm{ass}(S) \mid S \in \mathrm{Den}_l(R)\}$ where $\mathrm{ass}(S) := \{r \in R \mid sr = 0 \text{ for some } s = s(r) \in S\}$;
- $\mathrm{Den}_l(R, \mathfrak{a})$ is the set of left denominator sets S of R with $\mathrm{ass}(S) = \mathfrak{a}$ where $\mathfrak{a}$ is an ideal of R;
- $S_\mathfrak{a} = S_\mathfrak{a}(R) = S_{l,\mathfrak{a}}(R)$ is the *largest element* of the poset $(\mathrm{Den}_l(R, \mathfrak{a}), \subseteq)$ and $Q_\mathfrak{a}(R) := Q_{l,\mathfrak{a}}(R) := S_\mathfrak{a}^{-1}R$ is the *largest left quotient ring associated with* $\mathfrak{a}$. The fact that $S_\mathfrak{a}$ exists is proven in [3, Theorem 2.1] (but also see Lemma 2.5 below for the easy proof in other contexts);
- In particular, $S_0 = S_0(R) = S_{l,0}(R)$ is the largest element of the poset $(\mathrm{Den}_l(R, 0), \subseteq)$, i.e. the *largest regular left Ore set* of R, and $Q_l(R) := S_0^{-1}R$ is the *largest left quotient ring* of R [3];
- $\mathrm{max.Den}_l(R)$ is the set of maximal left denominator sets of R (it is always a *non-empty* set, see [3], or Lemma 2.5 below for the proof).

The largest strong left quotient ring of a ring. Consider the following subsets of a ring R: The sets

$$\mathcal{L}_l^s(R) := \bigcap_{S \in \mathrm{max.Den}_l(R)} S \overset{\text{Proposition 2.3.(1)}}{=} \{c \in R \mid \frac{c}{1} \in (S^{-1}R)^* \quad \text{for all } S \in \mathrm{max.Den}_l(R)\},$$