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Lie type algebras with an automorphism of finite order

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ABSTRACT

An algebra L over a field $\mathbb{F}$, in which product is denoted by $[\,,\,]$, is called a *Lie type algebra* if for all elements $a, b, c \in L$ there exist $\alpha, \beta \in \mathbb{F}$ (depending on a, b, c) such that $\alpha \neq 0$ and $[[a, b], c] = \alpha[a, [b, c]] + \beta[[a, c], b]$. Examples of Lie type algebras include associative algebras, Lie algebras, Leibniz algebras, etc. It is proved that if a Lie type algebra L admits an automorphism of finite order n with finite-dimensional fixed-point subalgebra of dimension m, then L has a soluble ideal of finite codimension bounded in terms of n and m and of derived length bounded in terms of n.

1. Introduction

By Kreknin's theorem [3] a Lie algebra over a field admitting a fixed-point-free automorphism of finite order n is soluble of derived length at most $\leqslant 2^n - 2$. In [7,10] it

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was proved that a Lie algebra with an "almost fixed-point-free" automorphism of finite order is almost soluble: if a Lie algebra L over a field admits an automorphism φ of finite order n such that the fixed-point subalgebra $C_L(\varphi)$ has finite dimension m, then L has a soluble ideal of finite codimension bounded in terms of n and m and of derived length bounded in terms of n.

The proofs of the above results are purely combinatorial and do not use the structure theory. This fact makes it possible to extend them to a broader class of algebras including associative algebras, Lie algebras, Leibniz algebras and others. Throughout the present paper, a *Lie type algebra* means an algebra L over a field $\mathbb{F}$ with product $[\,,\,]$ satisfying the following property: for all elements $a, b, c \in L$ there exist $\alpha, \beta \in \mathbb{F}$ such that $\alpha \neq 0$ and

$$[[a,b],c] = \alpha[a,[b,c]] + \beta[[a,c],b].$$

Note that in general α, β depend on elements $a, b, c \in L$; they can be viewed as functions $\alpha, \beta : L \times L \times L \to \mathbb{F}$.

The main result of the paper is the following

Theorem 1.1. *Suppose that a Lie type algebra L (of possibly infinite dimension) over an arbitrary field admits an automorphism of finite order n with finite-dimensional fixed-point subalgebra of dimension m. Then L has a soluble ideal of finite codimension bounded in terms of n and m and of derived length bounded in terms of n.*

Theorem 1.1 is also non-trivial for finite-dimensional Lie type algebras because of the bound for the codimension. No result of this kind is possible for an automorphism of *infinite* order: a free Lie algebra on the free generators f_i, $i \in \mathbb{Z}$, admits the fixed-point-free automorphism given by the mapping $f_i \to f_{i+1}$. The proof reduces to considering a $(\mathbb{Z}/n\mathbb{Z})$-graded algebra with finite-dimensional zero component (Theorem 1.2). Recall that an algebra L over a field with product $[\,,\,]$ is $(\mathbb{Z}/n\mathbb{Z})$-*graded* if

$$L = \bigoplus_{i=0}^{n-1} L_i \qquad \text{and} \qquad [L_i, L_j] \subseteq L_{i+j\,(\mathrm{mod}\, n)},$$

where L_i are subspaces of L. Elements of L_i are referred to as homogeneous and the subspaces L_i are called homogeneous components or grading components. In particular, L_0 is called the zero component.

Finite cyclic gradings naturally arise in the study of algebras admitting an automorphism of finite order. This is due to the fact that, after the ground field is extended by a primitive nth root of unity ω, the eigenspaces $L_j = \{a \in L \mid \varphi(a) = \omega^j a\}$ behave like the components of a $(\mathbb{Z}/n\mathbb{Z})$-grading: $[L_s, L_t] \subseteq L_{s+t}$, where $s+t$ is calculated modulo n. For example, Kreknin's theorem [3] can be reformulated in terms of graded Lie algebras as follows: a $(\mathbb{Z}/n\mathbb{Z})$-graded Lie algebra $L = \bigoplus_{i=0}^{n-1} L_i$ over an arbitrary field with trivial zero component $L_0 = 0$ is soluble of derived length at most $\leqslant 2^n - 2$. The proof of the