Journal of Algebra 439 (2015) 67–109

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Duality for generalised differentials on quantum groups

Shahn Majid, Wen-Qing Tao [1]

Queen Mary University of London, School of Mathematical Sciences, Mile End Rd, London E1 4NS, UK

ARTICLE INFO

Article history:
Received 12 September 2014
Available online 26 May 2015
Communicated by J.T. Stafford

MSC:
primary 81R50, 58B32, 20D05

Keywords:
Noncommutative geometry
Quantum group
Hopf algebra
Differential calculus
Bicovariant
Quiver
Shuffle algebra
Crossed module
Duality

ABSTRACT

We study generalised differential structures (Ω^1, d) on an algebra A, where $A \otimes A \to \Omega^1$ given by $a \otimes b \to a\mathrm{d}b$ need not be surjective. The finite set case corresponds to quivers with embedded digraphs, the Hopf algebra left covariant case to pairs (Λ^1, ω) where Λ^1 is a right module and ω a right module map, and the Hopf algebra bicovariant case corresponds to morphisms $\omega : A^+ \to \Lambda^1$ in the category of right crossed (or Drinfeld–Radford–Yetter) modules over A. When $A = U(\mathfrak{g})$ the generalised left covariant differential structures are classified by cocycles $\omega \in Z^1(\mathfrak{g}, \Lambda^1)$. We then introduce and study the dual notion of a codifferential structure (Ω^1, i) on a coalgebra and for Hopf algebras the self-dual notion of a strongly bicovariant differential graded algebra (Ω, d) augmented by a codifferential i of degree -1. Here Ω is a graded super-Hopf algebra extending the Hopf algebra $\Omega^0 = A$ and, where applicable, the dual super-Hopf algebra gives the same structure on the dual Hopf algebra. Accordingly, group 1-cocycles correspond precisely to codifferential structures on algebraic groups and function algebras. Among general constructions, we show that first order data (Λ^1, ω) on a Hopf algebra A extends canonically to a strongly bicovariant differential graded algebra via the

E-mail addresses: s.majid@qmul.ac.uk (S. Majid), w.tao@qmul.ac.uk (W.-Q. Tao).

[1] The second author was supported by the China Scholarship Council.

http://dx.doi.org/10.1016/j.jalgebra.2015.03.023

braided super-shuffle algebra. The theory is also applied to quantum groups with $\Omega^1(C_q(G))$ dually paired to $\Omega^1(U_q(\mathfrak{g}))$.

1. Introduction

We recall that in noncommutative geometry a 'space' is replaced by a 'coordinate' algebra and we define the differential structure algebraically. However, whereas on $\mathbb{R}^n$ and other Lie groups there is a unique translation-invariant calculus and this tends to be transferred throughout geometry, uniqueness is not the case in noncommutative geometry and this leads to a genuine degree of freedom. This can be formulated as a differential algebra, meaning an algebra A equipped with an A–A bimodule Ω^1 and an 'exterior derivative' $\mathrm{d}: A \to \Omega^1$ obeying the Leibniz rule

$$\mathrm{d}(ab) = (\mathrm{d}a)b + a\mathrm{d}b, \quad \forall a, b \in A$$

along with a 'surjectivity axiom' that $\phi : A \otimes A \to \Omega^1$, $a \otimes b \mapsto a\mathrm{d}b$ is surjective. One says that the differential calculus is connected if $\ker \mathrm{d} = k1$ where k is the ground field. The surjectivity axiom ensures that any calculus is a quotient of the universal one $\Omega^1_{univ} = \ker(m : A \otimes A \to A)$ by a sub-bimodule. This is because the above map remains surjective when restricted to Ω^1_{univ} and then becomes a bimodule map. One is also interested in extending Ω^1 to a differential graded algebra $\Omega = \oplus_{i\geq 0}\Omega^i$ with $\Omega^0 = A$ and d extending to a degree 1 super-derivation such that $\mathrm{d}^2 = 0$. The cohomology of this complex could be viewed as the 'noncommutative de Rham cohomology' of the differential algebra A and its extension (although this term is also used for other more specific constructions). The use of differential graded algebras goes back to Quillen and others in the 1970s and is now common to most approaches to noncommutative geometry.

In spite of its successes, this theory is unnecessarily restrictive and we now consider a natural generalisation where the surjectivity is dropped. There turn out to be many natural situations where this occurs and where we still have a differential complex of interest. One still has a standard differential calculus $\bar{\Omega}^1 \subseteq \Omega^1$ defined as the image of ϕ and our interest is in what happens to the rest of Ω^1 particularly when we extend to Ω. Section 2 covers this general theory at first order level including the example of A the algebra of functions on a finite set X. Here standard calculi correspond to digraphs on X while our generalised ones are given by quivers containing digraphs. Section 2 also generalises the theory of bicovariant differential calculi on Hopf algebras A which has been around for more than 20 years now [17]. We recall that standard bicovariant calculi correspond to right Ad-coaction stable right ideals in the augmentation ideal A^+ (the latter is the kernel of the counit of the Hopf algebra). The case of generalised bicovariant first order differential calculi is similar but consists in the richer data (Λ^1, ω), where