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Abelian quotients of triangulated categories

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ABSTRACT

We study abelian quotient categories $\mathcal{A} = \mathcal{T}/\mathcal{J}$, where $\mathcal{T}$ is a triangulated category and $\mathcal{J}$ is an ideal of $\mathcal{T}$. Under the assumption that the quotient functor is cohomological we show that it is representable and give an explicit description of the functor. We give technical criteria for when a representable functor is a quotient functor, and a criterion for when $\mathcal{J}$ gives rise to a cluster-tilting subcategory of $\mathcal{T}$. We show that the quotient functor preserves the AR-structure. As an application we show that if $\mathcal{T}$ is a finite 2-Calabi–Yau category, then with very few exceptions $\mathcal{J}$ is a cluster-tilting subcategory of $\mathcal{T}$.

1. Introduction

In the literature there are several known methods for forming a triangulated category given an abelian category. Given an abelian category $\mathcal{A}$ one can form the homotopy category $\mathcal{K}(\mathcal{A})$ and the derived category $\mathcal{D}(\mathcal{A})$, both of which are triangulated, along with their bounded versions. Orbit categories $\mathcal{D}^b(\mathcal{A})/F$ are known [10] to be triangulated when $\mathcal{A}$ is hereditary and F is a suitable autoequivalence. The stable module category of a selfinjective algebra is also triangulated.

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With the introduction of cluster algebras [7] and cluster-tilting theory [4], cluster-tilting subcategories (or maximal 1-orthogonal subcategories) have been defined, see [8]. In [11], Koenig and Zhu show that the quotient of any triangulated category by a cluster-tilting subcategory is abelian. However not all triangulated categories contain a cluster-tilting subcategory, but they may still admit an abelian quotient (for an example, see [11]). It is also known that for the cluster categories of coherent sheaves on weighted projective lines it is possible to obtain an abelian quotient by factoring out morphisms, without any objects being sent to zero [3].

Consider the orbit category $\mathcal{D}^b(kQ)/\Sigma$, where Q is a Dynkin diagram and Σ is the suspension functor. This category has the same (finite) number of isomorphism classes of indecomposable objects as $\operatorname{mod} kQ$, but has a greater number of irreducible morphisms. This motivates us to find out if we can factor out an ideal to obtain an abelian category, possibly without sending any non-zero objects to zero. Both of the examples mentioned will be revisited in detail in Section 4.

Factoring out an ideal from the cluster category of a hereditary algebra has been studied [5]. All known abelian quotients of these cluster categories arise from factoring out cluster-tilting subcategories. We show that in the finite case these are in fact all possible abelian quotient categories.

In Section 2 we define some notation and show that in the finite case, if an abelian quotient category exists, it has enough projectives.

In Section 3 we study a quotient functor from a triangulated category to an abelian category with projective generator. We show that it is representable and naturally equivalent to an explicitly described functor.

Section 4 contains the main result:

Theorem 1. $\operatorname{Hom}_{\mathcal{T}}(T,-)$ *is a quotient functor from a triangulated category* $\mathcal{T}$ *if and only if the following two criteria are satisfied*

- **a:** *For all right minimal morphisms* $T_1 \to T_0$*, where* $T_0, T_1 \in \operatorname{add} T$*, all distinguished triangles* $T_1 \to T_0 \to X \xrightarrow{h} \Sigma T_1$ *satisfy* $\operatorname{Hom}_{\mathcal{T}}(T,h) = 0$.
- **b:** *For all indecomposable* T*-supported objects* X *there exists a distinguished triangle* $T_1 \to T_0 \to X \xrightarrow{h} \Sigma T_1$ *with* $T_1, T_0 \in \operatorname{add} T$ *and* $\operatorname{Hom}_{\mathcal{T}}(T,h) = 0$.

In Section 5 we show that if it exists, the AR-structure is preserved by the quotient functor.

In Section 6 we discuss the special case of triangulated categories with Calabi–Yau dimension 2. We show

Theorem 2. *Let* $\mathcal{T}$ *be a 2-CY connected triangulated category with finitely many isomorphism classes of indecomposable objects. If* T *is an object in* $\mathcal{T}$ *such that* $\operatorname{Hom}_{\mathcal{T}}(T,-) : \mathcal{T} \to \operatorname{mod}\Gamma$ *is full and dense, then* T *is either Schurian or* T *a 2-cluster-tilting object in* $\mathcal{T}$*.*