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Invariants of Specht modules

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A R T I C L E   I N F O

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A B S T R A C T

In [14] Hemmer conjectures that the module of fixed points for the symmetric group Σ_m of a Specht module for Σ_n (with $n > m$), over a field of positive characteristic p, has a Specht series, when viewed as a Σ_{n-m}-module. We provide a counterexample for each prime p. The examples have the same form for $p \geq 5$ and we treat the cases $p = 3$ and $p = 2$ separately.

Introduction

Let k be a field of positive characteristic. For a positive integer n we denote by Σ_n the symmetric group of degree n. Let m, n be positive integers with $m < n$. The space X^{Σ_m} of points fixed by Σ_m of a module X over $k\Sigma_n$ is naturally a module for $k\Sigma_{n-m}$. It is conjectured in [14, Conjecture 7.2] (see also [15, Conjecture 7.3]) that this fixed point module has a Specht filtration. In case $m < p$ this follows by the exactness of the Σ_m fixed point functor. Here we study the case $m = p$ and provide counterexamples of Hemmer's Conjecture. The layout of the paper is the following. Section 1 is preliminary: we establish notation and remind the reader of connections between the representation

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theory of the general linear groups and symmetric groups. We also outline the strategy for finding invariants which is pursued in what follows. In Sections 2 and 3 we consider the case $p \geq 5$. The main purpose of Section 2 is to establish the dimensions of invariants for certain three part partitions with first part at most p. This is then used in Section 3 to show that the Specht module corresponding to the partition (p, p, p) is a counterexample. In Section 4 we carry out a similar analysis in the case $p = 3$ and show that the Specht module corresponding to the partition $(4, 4, 4)$ is a counterexample. In Section 4 we carry out a similar analysis in the case $p = 2$ and show that the Specht module corresponding to the partition $(4, 4)$ is a counterexample. In Sections 2 and 3 our arguments are of a general nature. In Sections 4 and 5, as well as general arguments, we found it convenient to use specific results on the nature of Specht modules to be found in (or easily deducible from) James's tables, [17, Appendix].

1. Preliminaries

We recall certain combinatorics associated to partitions, the usual set-up for the representation theory of general linear groups and symmetric groups and also some well-known results. We then describe some results that will be particularly useful to us. We conclude this section by describing the methodology to be followed in the rest of the paper.

1.1. Combinatorics

By a partition we mean an infinite sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of nonnegative integers with $\lambda_1 \geq \lambda_2 \geq \ldots$ and $\lambda_j = 0$ for j sufficiently large. If m is a positive integer such that $\lambda_j = 0$ for $j > m$ we identify λ with the finite sequence $(\lambda_1, \ldots, \lambda_m)$. The length $\mathrm{len}(\lambda)$ of a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ is 0 if $\lambda = 0$ and is the positive integer m such that $\lambda_m \neq 0$, $\lambda_{m+1} = 0$, if $\lambda \neq 0$. For a partition λ, we denote by λ' the transpose partition of λ. We write $\mathcal{P}$ for the set of partitions. We define the degree of a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ by $\deg(\lambda) = \lambda_1 + \lambda_2 + \cdots$.

We set $X(n) = \mathbb{Z}^n$. There is a natural partial order on $X(n)$. For $\lambda = (\lambda_1, \ldots, \lambda_n), \mu = (\mu_1, \ldots, \mu_n) \in X(n)$, we write $\lambda \leq \mu$ if $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$ for $i = 1, 2, \ldots, n-1$ and $\lambda_1 + \cdots + \lambda_n = \mu_1 + \cdots + \mu_n$. We shall use the standard $\mathbb{Z}$-basis $\epsilon_1, \ldots, \epsilon_n$ of $X(n)$, so $\epsilon_i = (0, \ldots, 1, \ldots, 0)$ (with 1 in the ith position). We write $\Lambda(n)$ for the set of n-tuples of nonnegative integers.

We write $X^+(n)$ for the set of dominant n-tuples of integers, i.e., the set of elements $\lambda = (\lambda_1, \ldots, \lambda_n) \in X(n)$ such that $\lambda_1 \geq \cdots \geq \lambda_n$. We write $\Lambda^+(n)$ for the set of partitions into at most n-parts, i.e., $\Lambda^+(n) = X^+(n) \bigcap \Lambda(n)$. We shall sometimes refer to elements of $\Lambda(n)$ as polynomial weights and elements of $\Lambda^+(n)$ as polynomial dominant weights. For a nonnegative integer r we write $\Lambda^+(n, r)$ for the set of partitions of r into at most n parts, i.e., the set of elements of $\Lambda^+(n)$ of degree r.