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The one-dimensional line scheme of a certain family of quantum $\mathbb{P}^3$s ☆

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A B S T R A C T

A quantum $\mathbb{P}^3$ is a noncommutative analogue of a polynomial ring on four variables, and, herein, it is taken to be a regular algebra of global dimension four. It is well known that if a generic quadratic quantum $\mathbb{P}^3$ exists, then it has a point scheme consisting of exactly twenty distinct points and a one-dimensional line scheme. In this article, we compute the line scheme of a family of algebras whose generic member is a candidate for a generic quadratic quantum $\mathbb{P}^3$. We find that, as a closed subscheme of $\mathbb{P}^5$, the line scheme of the generic member is the union of seven curves; namely, a nonplanar elliptic curve in a $\mathbb{P}^3$, four planar elliptic curves and two nonsingular conics.

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Introduction

A regular algebra of global dimension n is often viewed as a noncommutative analogue of a polynomial ring on n variables. Generalizing the language in [1], such an algebra is sometimes called a quantum $\mathbb{P}^{n-1}$. In [2], quantum $\mathbb{P}^2$s were classified according to their point schemes, with the point scheme of the most generic quadratic quantum $\mathbb{P}^2$ depicted by an elliptic curve in $\mathbb{P}^2$.

Consequently, a similar description is desired for quadratic quantum $\mathbb{P}^3$s using their point schemes or their line schemes, where the definition of line scheme was given in [11]. However, to date, very few line schemes of quadratic quantum $\mathbb{P}^3$s are known, especially of algebras that are candidates for generic quadratic quantum $\mathbb{P}^3$s. As explained in [14], if a generic quadratic quantum $\mathbb{P}^3$ exists, then it has a point scheme consisting of exactly twenty distinct points and a one-dimensional line scheme. Hence, in this article, we compute the line scheme of a family of algebras that appeared in [3, §5], and whose generic member is a candidate for a generic quadratic quantum $\mathbb{P}^3$.

The article is outlined as follows. Section 1 begins with some definitions, including the introduction of the family of algebras considered herein. The point schemes of the algebras are computed in Section 2 in Proposition 2.2, whereas Sections 3 and 4 are devoted to the computation of the line scheme and identifying the lines in $\mathbb{P}^3$ to which the points of the line scheme correspond. In particular, our main results are Theorems 3.1, 3.3 and 4.1. In the first two, we prove that the line scheme of the generic member is the union of seven curves; namely, a nonplanar elliptic curve in a $\mathbb{P}^3$ (a spatial elliptic curve), four planar elliptic curves and two nonsingular conics. In Theorem 4.1, we find that if p is one of the generic points of the point scheme, then there are exactly six distinct lines of the line scheme that pass through p. Appendix A lists polynomials that are used throughout the article.

It is hoped that data from the one-dimensional line scheme of any potentially generic quadratic quantum $\mathbb{P}^3$ will motivate conjectures and future research in the subject. In fact, the results herein suggest that the line scheme of the most generic quadratic quantum $\mathbb{P}^3$ is conceivably the union of two spatial elliptic curves and four planar elliptic curves (see Conjecture 4.2).

1. The algebras

In this section, we introduce the algebras from [3, §5] that are considered in this article.

Throughout the article, $\Bbbk$ denotes an algebraically closed field and $M(n, \Bbbk)$ denotes the vector space of $n \times n$ matrices with entries in $\Bbbk$. If V is a vector space, then $V^\times$ will denote the nonzero elements in V, and V^* will denote the vector-space dual of V. In this section, we take $\operatorname{char}(\Bbbk) \neq 2$, but, in Sections 3 and 4, we assume $\operatorname{char}(\Bbbk) = 0$ owing to the computations in those sections.