Journal of Algebra 439 (2015) 334–359

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On local and global conjugacy

Song Wang [1]

Academy of Mathematics and Systematics Sciences, the Morningside Center of Mathematics, the Key Laboratory of Hua, Chinese Academy of Science, China

ARTICLE INFO

Article history:
Received 17 January 2013
Available online 10 June 2015
Communicated by Michel Broué

Keywords:
Local conjugacy
Global conjugacy
LFMO-special representations
Multiplicity

ABSTRACT

Local and global conjugacy issue is closely related to the multiplicity one property in representation theory and Langlands' program. We classify all even dimensional orthogonal irreducible representations which are locally conjugate but not globally conjugate in image in SO($2N$), and with certain Langlands' functoriality these will lead to connected instances for the failure of multiplicity one for SO($2N$) (named as LFMO). In this note, we gather most our local–global results in a purely representation theoretic way.

Contents

E-mail address: songw1973@amss.ac.cn.

[1] The author was supported in part by One Hundred Talents Program at CAS, and national 973 project 2013CB834202.

http://dx.doi.org/10.1016/j.jalgebra.2015.05.001
0021-8693/

1. Introduction

Letting G be a reductive linear algebraic group over $\mathbb{C}$ and H be a closed subgroup of G, one may ask to what extent from the representation theory we can determine H. For example, the dimension datum for H in G consists of integers $m_H(\rho)$ where ρ runs through all finite dimensional representations of G, where $m_H(\rho)$ is the multiplicity of the trivial representation in the restriction of ρ to H. One question arises: does the dimension datum determine the isomorphism class and the conjugacy class of H? In this note, we discuss and classify a special family of counter-examples (called LFMO-special representation, which will be defined explicitly later) for $G = \mathrm{SO}(2N)$ and H connected reductive, and give negative answer to Langlands' question in some sense [15] (also see [16]). In fact, our first family of such examples (Theorem 3.14, also see [27] and [28]) gave first connected instances of locally conjugate subgroups of $G = \mathrm{SO}(2N)$ failing to be conjugate. Moreover, with sufficient functoriality, such counter-examples will give failure of multiplicity one, and thus got some attention in the study of beyond endoscopy which is much hotter after the establishment of the fundamental lemma by Ngo [11].

In 1990, M. Larsen and R. Pink studied the case $G = \mathrm{GL}(n)$, and got some results on dimension datum [20]. In fact, they proved, for general G and connected semisimple H, that the isomorphism class of H is determined by the dimension datum, and when $G = \mathrm{GL}(n)$ and H embeds into G in an essential way, i.e., irreducibly as an n-dimensional representation, the conjugacy class of H is determined by the dimension datum.[2] For general G, this is not the case. In fact, earlier works exhibited various counter-examples [5,18,19,27,1,29,28].

The discrepancy between the representation feature and the conjugacy occurs due to various reasons, and one is the "local–global issue". Let G be as above and H, H' be two closed subgroups of G. We say that H and H' are *locally conjugate* or *element-wise conjugate* if there is an isomorphism $i : H \xrightarrow{\cong} H'$ such that for $h \in H$ in a (Zariski) dense subset of H, h and $i(h)$ are conjugate. We say that H and H' are *globally conjugate* if they are conjugate in G. Also, there are definitions in terms of group homomorphisms. Let ρ, ρ': $H \to G$ be two homomorphisms of linear algebraic groups. We say that ρ and ρ' are *locally conjugate* if for $h \in H$ in a (Zariski) dense subset of H, $\rho(h)$ and $\rho'(h)$ are conjugate in G. We say that ρ and ρ' are *globally conjugate* if they are conjugate, namely, there exists a $g \in G$ such that $\rho'(h) = g\rho(h)g^{-1}$ for all h in H. We say that ρ and ρ' are *globally conjugate in image* if $\rho(H)$ and $\rho'(H)$ are conjugate in G.

[2] But for general H in $\mathrm{GL}(n)$, the conjugacy of H is not completely determined by the dimension datum; see [20,1,29].