Journal of Algebra 439 (2015) 360–372

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Singularities and holonomicity of binomial *D*-modules

Christine Berkesch Zamaere [a,*,1], Laura Felicia Matusevich [b,2], Uli Walther [c,3]

[a] *School of Mathematics, University of Minnesota, Minneapolis, MN 55455, United States*
[b] *Department of Mathematics, Texas A&M University, College Station, TX 77843, United States*
[c] *Department of Mathematics, Purdue University, West Lafayette, IN 47907, United States*

A R T I C L E   I N F O

Article history:
Received 5 January 2015
Available online 11 June 2015
Communicated by Luchezar L. Avramov

MSC:
primary 32C38, 14B05
secondary 33C70, 14M25

Keywords:
Binomial D-module
Holonomic
Singularity
A-discriminant
Hypergeometric

A B S T R A C T

We study binomial D-modules, which generalize A-hypergeometric systems. We determine explicitly their singular loci and provide three characterizations of their holonomicity. The first of these is an equivalence of holonomicity and L-holonomicity for these systems. The second refines the first by giving more detailed information about the L-characteristic variety of a non-holonomic binomial D-module. The final characterization states that a binomial D-module is holonomic if and only if its corresponding singular locus is proper.

* Corresponding author.
E-mail addresses: cberkesc@math.umn.edu (C. Berkesch Zamaere), laura@math.tamu.edu (L.F. Matusevich), walther@math.purdue.edu (U. Walther).

[1] C.B.Z. was partially supported by NSF Grants DMS 1303083, DMS 0901123, and OISE 0964985.
[2] L.F.M. was partially supported by NSF Grants DMS 0703866 and DMS 1001763 and a Sloan Research Fellowship.
[3] U.W. was partially supported by NSF Grants DMS 0901123 and DMS 1401392.

http://dx.doi.org/10.1016/j.jalgebra.2015.04.030
0021-8693/

1. Introduction

Binomial ideals in a polynomial ring over a field enjoy many special properties that set them apart from more general ideals. For example, work of Eisenbud and Sturmfels [8] shows that toric ideals can be viewed as basic building blocks of binomial ideals. Extending this point of view to D-modules, binomial D-modules (Definition 2.3) were introduced in [7] as a generalized framework to study systems of hypergeometric differential equations; here, the pendant to the toric ideals are the A-hypergeometric differential equations of Gelfand, Graev, Kapranov and Zelevinsky [9,10,12,13].

As in the polynomial case, binomial D-modules have some unusual properties. For instance, a binomial D-module is holonomic if and only if it has a finite dimensional solution space; while the forward implication in the previous statement is true in general, the converse certainly is not.

The goal of this article is to provide more results in this vein, further showing how special binomial D-modules are within the class of all D-modules. For this purpose, we study the characteristic variety and singular locus of a binomial D-module, and use our conclusions to obtain new characterizations of holonomicity for these objects.

Our main result is that a binomial D-module on $\mathbb{C}^n$ is holonomic if and only if its restriction to $(\mathbb{C}^*)^n$ is holonomic (Theorem 3.1), if and only if its singular locus is a proper subvariety of $\mathbb{C}^n$ (Theorem 4.2). As before, the forward implications are always true, but the converses fail in general, even in the simplest instances (Examples 3.3 and 4.3).

A strong motivation for the statements in this paper comes from our companion article [2]. In that work, the results here are used to obtain conclusions about classical systems of hypergeometric differential equations; see Remark 3.2 for more details.

1.1. Outline

In Section 2, we introduce concepts and notation about D-modules that will be used throughout. In Section 3, we prove Theorem 3.1 using ideas from [20]. In Section 4, we prove Theorem 4.2 by referring to a result from [2] and we give a combinatorial description of the singular locus of a binomial D-module along the lines of previous work by Gelfand, Kapranov and Zelevinsky, and Adolphson.

2. Preliminaries

2.1. Set-up

We let $d \leq n$ stand for two elements of the set of natural numbers $\mathbb{N} = 0, 1, 2, \ldots$.

Convention 2.1. Throughout this article, $A = [\, a_1\, a_2 \cdots a_n \,]$ is an integer $d \times n$ matrix such that $\mathbb{Z} \cdot A = \mathbb{Z}^d$ as lattices, and that there exists $h \in \mathbb{Q}^d$ such that $h \cdot a_i > 0$ for $i = 1, \ldots, n$.