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Projective modules for the symmetric group and Young's seminormal form

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ABSTRACT

We study the representation theory of the symmetric group $\mathfrak{S}_n$ in positive characteristic p. Using features of the LLT-algorithm we give a conjectural description of the projective cover $P(\lambda)$ of the simple module $D(\lambda)$ where λ is a p-restricted partition such that all ladders of the corresponding ladder partition are of order less than p. Inspired by the recent theory of KLR-algebras, we explain an algorithm that allows us to verify this conjectural description for $n \leq 15$, at least.

1. Introduction

Computing the decomposition matrices of the symmetric groups is one of the big open problems in representation theory. Until recently, the work in this area was guided by the James' conjecture which says that the decomposition matrices should coincide with those for the Hecke algebras of type A at roots of unity, for the prime in a certain range. In particular, the conjecture predicts the decomposition numbers to be given by certain Kazhdan–Lusztig polynomials.

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Two of the recent major developments in representation theory, Brundan and Kleshchev's isomorphism Theorem together with Elias and Williamson's algebraic proof of Soergel's conjecture, could have be seen as evidence in favor of the James' conjecture. But even more recently, Williamson changed the subject dramatically by publishing a preprint that gave counterexamples to the James' conjecture, see [28].

In this paper we combine the classical theory of Young's seminormal form, the Lascoux, Leclerc and Thibon algorithm and Brundan and Kleshchev's isomorphism Theorem to make the $\mathfrak{S}_n$-representation theory look formally like the representation theory of Soergel bimodules. This opens up a new perspective on the representation theory of $\mathfrak{S}_n$ which we believe will be fruitful in the future.

Let us explain the contents of the paper in more detail. Let p be a prime and let $\mathbb{F}_p$ be the field of p elements. Denote by $\mathrm{Par}_{res,n}$ the set of p-restricted partitions of n. As is well known, it parametrizes the simple $\mathbb{F}_p\mathfrak{S}_n$-modules so that we for $\lambda \in \mathrm{Par}_{res,n}$ have a simple $\mathbb{F}_p\mathfrak{S}_n$-module $D(\lambda)$ together with its projective cover $P(\lambda)$.

After Section 2, which is devoted to setting up the notation, we construct in Section 3 an idempotent $\widetilde{e}_\lambda \in \mathbb{F}_p\mathfrak{S}_n$ that plays an important role throughout the paper. The main ingredient for $\widetilde{e}_\lambda$ is Murphy's tableau class idempotent for the ladder class of λ.

Defining $\widetilde{A(\lambda)} := \mathbb{F}_p\mathfrak{S}_n\widetilde{e}_\lambda$ we obtain a projective $\mathbb{F}_p\mathfrak{S}_n$-module, but it is decomposable in general, that is $\widetilde{A(\lambda)} \neq P(\lambda)$. On the other hand, we show in Theorem 3 that there is triangular expansion of the form

$$\widetilde{A(\lambda)} = P(\lambda) \oplus \bigoplus_{\mu,\mu\rhd\lambda} P(\mu)^{\oplus m_{\lambda\mu}} \tag{1}$$

for certain nonnegative integers $m_{\lambda\mu}$ where $\rhd$ is the usual dominance order on partitions.

In Section 4 we consider the Lascoux, Leclerc and Thibon (LLT) algorithm which gives a way of calculating the global crystal basis $\{G(\lambda) \,|\, \lambda \in \mathrm{Par}_{res,n}\}$, for the basic submodule $\mathcal{M}_q$ of the q-Fock space. An important ingredient for this algorithm is given by certain elements $A(\lambda)$ of $\mathcal{M}_q$ that LLT called 'the first approximation of the global basis'. In fact, their algorithm is a triangular recursion based on these elements. We use this to observe that they satisfy the following triangular expansion property

$$A(\lambda) = G(\lambda) + \sum_{\mu,\mu\rhd\lambda} n_{\lambda\mu}(q)G(\mu)$$

where $n_{\lambda\mu}(q) \in \mathbb{Z}[q,q^{-1}]$.

Our main point is now to consider $A(\lambda)$ as an object of interest in itself, and not just a tool for calculating $G(\lambda)$. In this spirit we conjecture that $A(\lambda)$ should be categorified by $\widetilde{A(\lambda)}$, or to be more precise that we should have

$$n_{\lambda\mu}(1) = m_{\lambda\mu}. \tag{2}$$