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The connectedness of Hessenberg varieties

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In this paper we consider certain closed subvarieties of the flag variety, known as Hessenberg varieties. These varieties arise in representation theory, algebraic geometry, and combinatorics. We give a connectedness criterion for semisimple Hessenberg varieties that generalizes a criterion given by Anderson and Tymoczko. It also generalizes results of Iveson in type A which prove that all Hessenberg varieties satisfying this criterion are connected. We then show that nilpotent Hessenberg varieties are rationally connected.

1. Introduction

Hessenberg varieties are a family of subvarieties of the flag variety first introduced by de Mari, Procesi, and Shayman in [3]. These varieties are defined by two parameters; an element X of the Lie algebra $\mathfrak{g}$ and a subspace $H \subseteq \mathfrak{g}$ called a Hessenberg space. In [3], the authors show that Hessenberg varieties corresponding to a regular semisimple element S are smooth and paved by affines for any choice of Hessenberg space. They also show that when H takes a particular form the resulting Hessenberg variety is a toric variety. This toric variety gives a representation of the Weyl group on its cohomology (see [10,12]). In [15], Tymoczko generalizes this action to the (equivariant) cohomology

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of semisimple Hessenberg varieties. In type A, this representation has been connected to quasisymmetric functions by Shareshian and Wachs (see [11]).

If a semisimple Hessenberg variety is connected, then its corresponding moment graph is connected. These graphs are an important tool in the study of the representation discussed above. Results on the geometric structure of semisimple Hessenberg varieties enhance our understanding of the corresponding moment graphs and, by extension, this representation.

Nilpotent Hessenberg varieties are natural generalizations of Springer fibers. In fact, when H is a Borel subalgebra the resulting Hessenberg variety is a Springer fiber. Many other well-studied subvarieties of the flag variety also occur as examples of Hessenberg varieties including Schubert and Peterson varieties. Strengthening results on the structure theory of Hessenberg varieties will provide a new insight into the structure theory of these and generalize methods currently used to study them.

In this paper we explore the connectedness properties of semisimple and nilpotent Hessenberg varieties. We give a criterion for semisimple Hessenberg varieties to be connected and prove that nilpotent Hessenberg varieties are rationally connected.

Let G be a linear, reductive algebraic group over $\mathbb{C}$, B a Borel subgroup, and let $\mathfrak{g}$, $\mathfrak{b}$ denote their respective Lie algebras. A Hessenberg space H is a linear subspace of $\mathfrak{g}$ that contains $\mathfrak{b}$ and is closed under the Lie bracket with $\mathfrak{b}$. Fix an element $X \in \mathfrak{g}$ and a Hessenberg space H. The Hessenberg variety $\mathcal{B}(X, H)$ is the subvariety of the flag variety $G/B = \mathcal{B}$ consisting of all cosets gB such that $g^{-1} \cdot X \in H$ where $g \cdot X$ denotes the adjoint action $Ad(g)(X)$.

When $S \in \mathfrak{g}$ is semisimple, we say $\mathcal{B}(S, H)$ is semisimple. In [9] it is shown that all such Hessenberg varieties are paved by affines. Applying this result, we have a method of computing the Betti numbers of $\mathcal{B}(S, H)$. As a consequence, we are able to show the following, which is Theorem 3.4 below.

Theorem 1. *If $S \in \mathfrak{g}$ is a semisimple element which is not in the center of $\mathfrak{g}$, then $\mathcal{B}(S, H)$ is connected if and only if the Hessenberg space H contains all root space vectors corresponding to the negative simple roots.*

The proof depends entirely on combinatorial properties of the root system associated to $\mathfrak{b} \subset \mathfrak{g}$. This criterion was previously only known when $S \in \mathfrak{g}$ is a regular semisimple element (see [1, Appendix A] and [3, Corollary 9(i)]). In type A, Iveson proves any Hessenberg variety satisfying the criteria from Theorem 1 must be connected (see Corollary 3.6, Part 2 in [6]).

When $N \in \mathfrak{g}$ is nilpotent, we say that $\mathcal{B}(N, H)$ is nilpotent. When N is regular in a Levi, $\mathcal{B}(N, H)$ is paved by affines (see [9,13,14]). However, instead of using an affine paving to compute Betti numbers, we construct a sequence of rational curves connecting any two points in $\mathcal{B}(N, H)$. A variety is rationally connected if any two points can be connected by a chain of rational curves, therefore we have the following for any choice of nilpotent $N \in \mathfrak{g}$.