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Relative dimension of morphisms and dimension for algebraic stacks

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ABSTRACT

Motivated by applications in moduli theory, we introduce a theory of lower bounds on relative dimension of morphisms of schemes and algebraic stacks. The theory is robust, applies to a wide range of situations, and has strong consequences. We thus obtain simple tools for making dimension-based deformation arguments on moduli spaces. In a complementary direction, we develop the basic properties of codimension for algebraic stacks.

1. Introduction

The notion of dimension for schemes is poorly behaved, even for relatively basic examples such as schemes smooth over the spectrum of a discrete valuation ring. Better behaved are codimension, and dimension of local rings. Thus, to translate naive dimension-based arguments from schemes of finite type over a field to a more general setting, a standard approach is to rephrase results using these alternatives. The purpose of the present paper is twofold: first, to introduce a more natural way of working with relative dimension of morphisms, and second, to generalize this – as well as basic properties of codimension – to algebraic stacks.

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In Definition 3.1 below, we give a precise formulation of what it should mean for a morphism to have relative dimension at least a given number n. Our immediate motivation is that in certain moduli space constructions in Brill–Noether theory (particularly the limit linear series spaces introduced by Eisenbud and Harris), a key property of the moduli space is that it has at least a certain dimension relative to the base. This allows the use of deformation arguments based purely on dimension counts. However, the existing language to express these ideas is notably lacking when the base is not of finite type over a field. Our definition gives very natural language to capture what is going on, and we show that it is formally well behaved, occurs frequently, and has strong consequences of the sort that one wants for moduli theory.

In generalizing limit linear series to higher-rank vector bundles, it is natural to work not with schemes but with algebraic stacks, and in this context, it is even more difficult to give transparent relative dimension statements using the usual tools. In particular, there is no good notion of dimension of local rings of stacks (see Example 6.9 below). Thus, the fact that our language generalizes readily to the stack context makes it especially useful, and it is incorporated accordingly into [12] and [13]. See Remark 1.1 for details. We emphasize however that the situation which arises for limit linear series is not very special, so we expect that it might arise in various other moduli problems as well, and our language would apply equally well to these cases.

Finally, we also develop the theory of codimension in the context of stacks, and relate it to relative dimension. Here, the statements are as expected, but we emphasize that the context of stacks introduces certain subtleties which demand a careful treatment. These arise in large part because of the tendency of smooth (and even étale) covers to break irreducible spaces into reducible ones. For instance, it is due to these phenomena that the condition of being universally catenary does not descend under étale morphisms, in general.

Remark 1.1. We describe in more detail how our ideas fit into the context of moduli problems. In the theory of limit linear series developed by Eisenbud and Harris, the main foundational theorem constructs a moduli space $G^r_d(X/B)$ for families of curves X/B, which parametrizes linear series over points corresponding to smooth curves, and limit linear series over points corresponding to singular curves. The key point is that $G^r_d(X/B)$ is cut out by a collection of Schubert conditions inside a space which is smooth over the base B, and thus, in our language, has universal relative dimension at least the classical Brill–Noether number ρ. It then follows that if the space of limit linear series over a particular curve X_0 of the family is nonempty of dimension ρ, then every limit linear series on X_0 is a limit of linear series on nearby smooth curves in the family.

Even in the scheme context, existing language was unsatisfactory to express what has been described above. The foundational theorem should be a statement about the relative dimension of $G^r_d(X/B)$ over B. However, as soon as B is not of finite type over a field (for instance, if B is the spectrum of a DVR), it need not follow from the construction that $\dim G^r_d(X/B) \geq \dim B + \rho$. One could express the foundational theorem in terms