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Primitive axial algebras of Jordan type

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A R T I C L E I N F O

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A B S T R A C T

An *axial algebra* over the field $\mathbb{F}$ is a commutative algebra generated by idempotents whose adjoint action has multiplicity-free minimal polynomial. For semisimple associative algebras this leads to sums of copies of $\mathbb{F}$. Here we consider the first nonassociative case, where adjoint minimal polynomials divide $(x-1)x(x-\eta)$ for fixed $0 \neq \eta \neq 1$. Jordan algebras arise when $\eta = \frac{1}{2}$, but our motivating examples are certain Griess algebras of vertex operator algebras and the related Majorana algebras. We study a class of algebras, including these, for which axial automorphisms like those defined by Miyamoto exist, and there classify the 2-generated examples. Always for $\eta \neq \frac{1}{2}$ and in identifiable cases for $\eta = \frac{1}{2}$ this implies that the Miyamoto involutions are 3-transpositions, leading to a classification.

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1. Introduction

Throughout we consider commutative $\mathbb{F}$-algebras A where $\mathbb{F}$ is a field. We emphasize that our algebras will usually be nonassociative and may not have an identity element.

For the element a of A and $\lambda \in \mathbb{F}$, the λ-eigenspace for the adjoint $\mathbb{F}$-endomorphism $\mathrm{ad}_a\colon x \mapsto xa$ of A will be denoted $A_\lambda(a)$ (where we allow the possibility $A_\lambda(a) = 0$). If A is an associative algebra and a is an idempotent element, then $A = A_1(a) \oplus A_0(a)$—the adjoint of the idempotent is semisimple with at most the two eigenvalues 0 and 1. Here we are interested in the minimal nonassociative case—semisimple idempotents whose adjoint eigenvalues are drawn from the set $\Lambda = \{1, 0, \eta\}$ for some $\eta \in \mathbb{F}$ with $0 \neq \eta \neq 1$.

An idempotent whose adjoint is semisimple will be called an *axis*. A commutative algebra generated by axes is then an *axial algebra*. The commutative algebra A over $\mathbb{F}$ is a *primitive axial algebra of Jordan type* η provided it is generated by a set of axes with each member a satisfying:

(a) $A = A_1(a) \oplus A_0(a) \oplus A_\eta(a)$.
(b) $A_1(a) = \mathbb{F}a$.
(c) $A_0(a)$ is a subalgebra of A.
(d) For all $\delta, \epsilon \in \pm$,

$$A_\delta(a)A_\epsilon(a) \subseteq A_{\delta\epsilon}(a)\,,$$

where $A_+(a) = A_1(a) \oplus A_0(a)$ and $A_-(a) = A_\eta(a)$.

Examples include Jordan algebras that are generated by idempotents [17]. These occur for $\eta = \frac{1}{2}$, although this is the case in which we say the least. Instead our motivation comes from the values $\eta = \frac{1}{4}$ and $\eta = \frac{1}{32}$, which arise as special cases of $\Lambda = \{1, 0, \frac{1}{4}, \frac{1}{32}\}$. Algebras of this latter type are provided by Griess algebras associated with vertex operator algebras and Majorana algebras [15,20,22,27].

A major accomplishment in the Griess algebra case was Sakuma's Theorem [27] which classified all 2-generated subalgebras. See also [15,16,12]. The following similar theorem is a central result of this paper.

(1.1). Theorem. *Let $\mathbb{F}$ be a field of characteristic not two with $\eta \in \mathbb{F}$ for $0 \neq \eta \neq 1$. Let A be a primitive axial $\mathbb{F}$-algebra of Jordan type η that is generated by two axes. Then we have one of the following:*

(1) *A is an algebra $\mathbb{F}$ of type* 1A *over $\mathbb{F}$;*
(2) *A is an algebra $\mathbb{F} \oplus \mathbb{F}$ of type* 2B *over $\mathbb{F}$;*
(3) *A is an algebra of type* $3\mathrm{C}(\eta)$ *of dimension* 3 *over $\mathbb{F}$;*
(4) *$\eta = -1$ and A is an algebra of type* $3\mathrm{C}(-1)^\times$ *of dimension* 2 *over $\mathbb{F}$;*