Journal of Algebra 437 (2015) 308–343

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Dual euclidean Artin groups and the failure of the lattice property

Jon McCammond [1]

Department of Mathematics, University of California, Santa Barbara, CA 93106, United States

ARTICLE INFO

Article history:
Received 13 November 2014
Available online 21 May 2015
Communicated by Patrick Dehornoy

MSC:
20F36

Keywords:
Euclidean Artin groups
Garside structures
Dual presentations
Lattices

ABSTRACT

The irreducible euclidean Coxeter groups that naturally act geometrically on euclidean space are classified by the well-known extended Dynkin diagrams and these diagrams also encode the modified presentations that define the irreducible euclidean Artin groups. These Artin groups have remained mysterious with some exceptions until very recently. Craig Squier clarified the structure of the three examples with three generators more than twenty years ago and François Digne more recently proved that two of the infinite families can be understood by constructing a dual presentation for each of these groups and showing that it forms an infinite-type Garside structure. In this article I establish that none of the remaining dual presentations for irreducible euclidean Artin groups correspond to Garside structures because their factorization posets fail to be lattices. These are the first known examples of Artin groups where all of their dual presentations fail to form Garside structures. Nevertheless, the results presented here about the cause of this failure form the foundation for a subsequent article in which the structure of euclidean Artin groups is finally clarified.

E-mail address: jon.mccammond@math.ucsb.edu.

[1] Support by the National Science Foundation is gratefully acknowledged (DMS-0805716).

http://dx.doi.org/10.1016/j.jalgebra.2015.04.021

There is an irreducible Artin group of euclidean type for each of the extended Dynkin diagrams. In particular, there are four infinite families, $\widetilde{A}_n$, $\widetilde{B}_n$, $\widetilde{C}_n$ and $\widetilde{D}_n$, known as the *classical* types plus five remaining *exceptional* examples $\widetilde{E}_8$, $\widetilde{E}_7$, $\widetilde{E}_6$, $\widetilde{F}_4$, and $\widetilde{G}_2$. Most of these groups have been poorly understood until very recently. Among the few known results are a clarification of the structure of the Artin groups of types $\widetilde{A}_2$, $\widetilde{C}_2$ and $\widetilde{G}_2$ by Craig Squier in [26] and two papers by François Digne [13,14] proving that the Artin groups of type $\widetilde{A}_n$ and $\widetilde{C}_n$ have dual presentations that are Garside structures. Our main result is that Squier's and Digne's examples are the only ones that have dual presentations that are Garside structures.

Theorem A *(Dual presentations and Garside structures). The unique dual presentation of* $\text{Art}(\widetilde{X}_n)$ *is a Garside structure when X is C or G and it is not a Garside structure when X is B, D, E or F. When $X = A$ there are distinct dual presentations of the group and Digne found that only one of them is a Garside structure.*

The proof is made possible by a simple combinatorial model developed in collaboration with Noel Brady that encodes all minimal length factorizations of a euclidean isometry into reflections [5]. Although the results here are essentially negative, they establish the foundations for positive results presented in [24]. In this subsequent article a new class of Garside groups are constructed from crystallographic groups closely related to euclidean Coxeter groups and these new groups contain euclidean Artin groups as subgroups, thereby clarifying their algebraic structure. For a survey of all three articles see [22].

The article is structured as follows. The first sections give basic definitions, define dual Artin groups and dual presentations, and review the results of [5] on factoring euclidean isometries into reflections. The middle sections apply these results to understand how Coxeter elements of irreducible euclidean Coxeter groups can be factored into reflections present in the group, thereby constructing dual presentations. The final sections record explicit results on a type-by-type basis, from which the main theorem immediately follows.

I would like to thank the anonymous referees for their close and detailed reading of the article. And, as a final note, I would like to highlight the fact that the rough outline of the main theorem was established in collaboration with John Crisp several years ago while I was visiting him in Dijon. John has since left mathematics but his central role in the genesis of this work needs to be acknowledged.

1. Basic definitions

This short section provides some basic definitions that are included for completeness. The terminology roughly follows [16,17] and [28].

Definition 1.1 *(Coxeter groups).* A *Coxeter group* is any group W that can be defined by a presentation of the following form. It has a standard finite generating set S and only