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Subgroups of polynomial automorphisms with diagonalizable fibers

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ABSTRACT

Let R be an integral domain over a field k, and G a subgroup of the automorphism group of the polynomial ring $R[x_1, \dots, x_n]$ over R. In this paper, we discuss when G is diagonalizable under the assumption that G is diagonalizable over the field of fractions of R. We are particularly interested in the case where G is a finite abelian group. Kraft and Russell (2014) [8] imply that every finite abelian subgroup of $\mathrm{Aut}_R R[x_1, x_2]$ is diagonalizable if R is an affine PID over $k = \mathbf{C}$. One of the main results of this paper says that the same holds for a PID R over any field k containing enough roots of unity.

1. Introduction

For each commutative ring R, we denote by $R[\mathbf{x}] = R[x_1, \dots, x_n]$ the polynomial ring in n variables $x_1, \dots, x_n$ over R, and by $\mathrm{Aut}_R R[\mathbf{x}]$ the automorphism group of the R-algebra $R[\mathbf{x}]$. We identify an endomorphism ϕ of the R-algebra $R[\mathbf{x}]$ with the n-tuple $(\phi(x_1), \dots, \phi(x_n))$ of elements of $R[\mathbf{x}]$, where the composition is defined by $\phi \circ \psi = (\phi(\psi(x_1)), \dots, \phi(\psi(x_n)))$. Note that, if G is a subgroup of $\mathrm{Aut}_R R[\mathbf{x}]$, and S is a

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commutative R-algebra, then $G_S := \{\mathrm{id}_S \otimes \phi \mid \phi \in G\}$ is a subgroup of $\mathrm{Aut}_S\, S[\mathbf{x}]$. When $S = \kappa(\mathfrak{p})$ is the residue field of the localization $R_\mathfrak{p}$ of R at a prime ideal $\mathfrak{p}$ of R, we denote G_S by $G_\mathfrak{p}$. If R is a domain, K always denotes the field of fractions of R.

Throughout this paper, let k be an arbitrary field. If R is a k-algebra, then $D_n(k) := \{\delta_\mathbf{a} \mid \mathbf{a} \in (k^*)^n\}$ is a subgroup of $\mathrm{Aut}_R\, R[\mathbf{x}]$, where we define $\delta_\mathbf{a} := (a_1x_1, \ldots, a_nx_n)$ for each $\mathbf{a} = (a_1, \ldots, a_n) \in (k^*)^n$. We say that a subgroup G of $\mathrm{Aut}_R\, R[\mathbf{x}]$ is *diagonalizable* if there exists $\psi \in \mathrm{Aut}_R\, R[\mathbf{x}]$ such that $\psi^{-1} \circ G \circ \psi$ is contained in $D_n(k)$.

Now, assume that R is a k-domain. In this paper, we discuss the following problems.

Problem 1. Let G be a subgroup of $\mathrm{Aut}_R\, R[\mathbf{x}]$ such that $G_{(0)}$ is diagonalizable. Does it follow that G is diagonalizable?

If we regard $\mathrm{Aut}_R\, R[\mathbf{x}]$ as a subgroup of $\mathrm{Aut}_K\, K[\mathbf{x}]$, then the assumption of Problem 1 is equivalent to $\psi^{-1} \circ G \circ \psi \subset D_n(k)$ for some $\psi \in \mathrm{Aut}_K\, K[\mathbf{x}]$. When $n = 2$, this condition implies that $G_\mathfrak{p}$ is diagonalizable for any prime ideal $\mathfrak{p}$ of R by van der Kulk [7] and Serre [14] (cf. Section 2 (c)). So we also consider the following problem for $n \geq 3$.

Problem 2. Let G be a subgroup of $\mathrm{Aut}_R\, R[\mathbf{x}]$ such that $G_\mathfrak{p}$ is diagonalizable for all the prime ideals $\mathfrak{p}$ of R. Does it follow that G is diagonalizable?

We are particularly interested in the case where G is a finite abelian group. In fact, whether every finite abelian subgroup of $\mathrm{Aut}_\mathbf{C}\, \mathbf{C}[\mathbf{x}]$ is conjugate to a subgroup of $D_n(\mathbf{C})$ is a difficult problem with little progress for $n \geq 3$ (see [5] for the case $n = 2$). This problem is a special case of Kambayashi's Linearization Problem [6], and is open even for finite cyclic groups (cf. [9]). In the case of finite cyclic groups, the problem is also included in the list of "eight challenging open problems in affine spaces" by Kraft [10]. We mention that, over a field of positive characteristic, a negative answer to a similar problem is already given by Asanuma [1]. It seems more difficult to diagonalize G in the case of positive characteristic than the other case.

Now, let us state our main result. Under the assumptions in Problems 1 and 2, there exists a subgroup $\mathcal{G}$ of $(k^*)^n$ for which $G_{(0)}$ is conjugate to $\{\delta_\mathbf{a} \mid \mathbf{a} \in \mathcal{G}\}$ in $\mathrm{Aut}_K\, K[\mathbf{x}]$. We write $\mathbf{a}^i := a_1^{i_1} \cdots a_n^{i_n}$ for each $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{G}$ and $i = (i_1, \ldots, i_n) \in \mathbf{Z}^n$, and define $M_\mathcal{G}$ to be the set of $i \in \mathbf{Z}^n$ such that $\mathbf{a}^i = 1$ for all $\mathbf{a} \in \mathcal{G}$. Let $\gamma_1, \ldots, \gamma_n$ be the images of the coordinate unit vectors of $\mathbf{Z}^n$ in $\Gamma_\mathcal{G} := \mathbf{Z}^n / M_\mathcal{G}$. For each i, let $\Gamma_\mathcal{G}^{(i)}$ be the subgroup of $\Gamma_\mathcal{G}$ generated by γ_j for $1 \leq j \leq n$ with $j \neq i$.

The following theorem is the main result of this paper.

Theorem 1.1. (i) *When $n = 2$, Problem 1 has an affirmative answer in the following two cases:*

(1) *R is a PID.*
(2) *R is a regular UFD, and $\Gamma_\mathcal{G}^{(1)}$ or $\Gamma_\mathcal{G}^{(2)}$ is not equal to $\Gamma_\mathcal{G}$.*