Journal of Algebra 435 (2015) 204–222

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Direct products of modules whose endomorphism rings have at most two maximal ideals

Adel Alahmadi [a], Alberto Facchini [b,*,1]

[a] *Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia*
[b] *Dipartimento di Matematica Università di Padova, 35121 Padova, Italy*

A R T I C L E   I N F O

Article history:
Received 11 March 2014
Available online 15 April 2015
Communicated by Louis Rowen

MSC:
16D70
16D80

Keywords:
Uniserial module
Monogeny class
Epigeny class
Direct-product decomposition

A B S T R A C T

Monogeny classes and epigeny classes have proved to be useful in the study of direct sums of uniserial modules and other classes of modules. In this paper, we show that they also turn out to be useful in the study of direct products.

1. Introduction

Two right R-modules M and N are said to *belong to the same monogeny class* (written $[M]_m = [N]_m$) if there exist a monomorphism $M \to N$ and a monomorphism $N \to M$.

* Corresponding author.
E-mail addresses: analahmadi@kau.edu.sa (A. Alahmadi), facchini@math.unipd.it (A. Facchini).

1 Partially supported by Università di Padova (Progetto ex 60% "Anelli e categorie di moduli") and Fondazione Cassa di Risparmio di Padova e Rovigo (Progetto di Eccellenza "Algebraic structures and their applications"), Italy.

http://dx.doi.org/10.1016/j.jalgebra.2015.04.002
0021-8693/

Dually, M and N are said to *belong to the same epigeny class* ($[M]_e = [N]_e$) if there exist an epimorphism $M \to N$ and an epimorphism $N \to M$. Recall that a module is *uniserial* if its lattice of submodules is linearly ordered under inclusion. In [7, Theorem 1.9], it was proved that if $U_1, \dots, U_n$, $V_1, \dots, V_t$ are non-zero uniserial right R-modules, then $U_1 \oplus \cdots \oplus U_n \cong V_1 \oplus \cdots \oplus V_t$ if and only if $n = t$ and there are two permutations σ, τ of $\{1, 2, \dots, n\}$ such that $[U_i]_m = [V_{\sigma(i)}]_m$ and $[U_i]_e = [V_{\tau(i)}]_e$ for every $i = 1, 2, \dots, n$. This result, which made possible the solution of a problem [23, p. 189] posed by Warfield in 1975, was then generalized in various directions. On the one hand, it was extended to the case of arbitrary, non-necessarily finite, families $\{\, U_i \mid i \in I \,\}$, $\{\, V_j \mid j \in J \,\}$ of uniserial modules [20, Theorem 2.6] (see Theorem 3.2 below). On the other hand, it was shown that similar theorems hold not only for uniserial modules, but also for cyclically presented modules over a local ring R, for kernels of morphisms between indecomposable injective modules, for couniformly presented modules, and more generally, for several classes of modules with at most two maximal right ideals (see [11, Section 5] and [15]).

In this paper, we prove that a similar result holds not only for direct sums, but also for direct products of arbitrary families $\{\, U_i \mid i \in I \,\}$, $\{\, V_j \mid j \in J \,\}$ of uniserial modules. We show (Theorem 3.1) that if there exist two bijections $\sigma, \tau \colon I \to J$ such that $[U_i]_m = [V_{\sigma(i)}]_m$ and $[U_i]_e = [V_{\tau(i)}]_e$ for every $i \in I$, then $\prod_{i \in I} U_i \cong \prod_{j \in J} V_j$. In fact, the theorem we prove is much more general, and involves completely prime ideals in categories of modules whose endomorphism rings have at most two maximal right ideals (Theorem 2.3). This allows us to apply our theorem not only to uniserial modules, but also to several other classes of modules, like the class of cyclically presented modules over a local ring and the class of kernels of morphisms between indecomposable injective modules (Section 3).

We then show with some examples that in general it is not possible to reverse our result (find the converse of it). It is possible to reverse it only in the particular case of slender modules (Theorems 5.3 and 5.5). For this class of modules, it is possible to argue as in the recent paper [16].

The rings we deal with are associative rings with identity $1 \neq 0$, and modules are unitary modules.

2. The main result

In order to present our result in the most general setting, that of modules whose endomorphism rings have at most two maximal right ideals, we adopt the point of view of [15, Section 6]. Thus, let R be an associative ring with identity and Mod-R the category of all right R-modules. Let $\mathcal{C}$ be a full subcategory of Mod-R whose class of objects $\mathrm{Ob}(\mathcal{C})$ consists of indecomposable right R-modules. Recall that a *completely prime ideal* $\mathcal{P}$ of $\mathcal{C}$ consists of a subgroup $\mathcal{P}(A, B)$ of the additive abelian group $\mathrm{Hom}_R(A, B)$ for every pair of objects $A, B \in \mathrm{Ob}(\mathcal{C})$ such that: