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On stable equivalences with endopermutation source

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ABSTRACT

We show that a bimodule between block algebras which has a fusion stable endopermutation module as a source and which induces Morita equivalences between centralisers of nontrivial subgroups of a defect group induces a stable equivalence of Morita type; this is the converse to a theorem of Puig. The special case where the source is trivial has long been known by many authors. The earliest instance of a result deducing a stable equivalence of Morita type from local Morita equivalences with possibly nontrivial endopermutation source is due to Puig, in the context of blocks with abelian defect groups with a Frobenius inertial quotient. The present note is motivated by an application, due to Biland, to blocks of finite groups with structural properties known to hold for hypothetical minimal counterexamples to the Z_p^*-Theorem.

1. Introduction

Let p be a prime and $\mathcal{O}$ a complete discrete valuation ring having a residue field k of characteristic p; we allow the case $\mathcal{O} = k$. We will assume that k is a splitting field for all block algebras which arise in this note. Following Broué [7, §5.A], given two $\mathcal{O}$-algebras A, B, an A-B-bimodule M and a B-A-bimodule N, we say that *M and N induce a stable equivalence of Morita type between A and B* if M, N are finitely generated projective as

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left and right modules, and if $M \otimes_B N \cong A \oplus W$ for some projective $A \otimes_{\mathcal{O}} A^{\mathrm{op}}$-module W and $N \otimes_A M \cong B \oplus W'$ for some projective $B \otimes_{\mathcal{O}} B^{\mathrm{op}}$-module W'. By a result of Puig in [25, 7.7.4] a stable equivalence of Morita type between block algebras of finite groups given by a bimodule with endopermutation source and its dual implies that there is a canonical identification of the defect groups of the two blocks such that both have the same local structure and such that corresponding blocks of centralisers of nontrivial subgroups of that common defect group are Morita equivalent via bimodules with endopermutation sources. The following theorem is the converse to this result. The terminology and required background information for this statement are collected in the next two sections, together with further references.

Theorem 1.1. *Let A, B be almost source algebras of blocks of finite group algebras over $\mathcal{O}$ having a common defect group P and the same fusion system $\mathcal{F}$ on P. Let V be an $\mathcal{F}$-stable indecomposable endopermutation $\mathcal{O}P$-module with vertex P, viewed as an $\mathcal{O}\Delta P$-module through the canonical isomorphism $\Delta P \cong P$. Let M be an indecomposable direct summand of the A-B-bimodule*

$$A \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta P}^{P\times P}(V) \otimes_{\mathcal{O}P} B\ .$$

Suppose that $(M \otimes_B M^*)(\Delta P) \neq \{0\}$. Then for any nontrivial fully $\mathcal{F}$-centralised subgroup Q of P, there is a canonical $A(\Delta Q)$-$B(\Delta Q)$-bimodule M_Q satisfying $\mathrm{End}_k(M_Q) \cong (\mathrm{End}_{\mathcal{O}}(M))(\Delta Q)$. Moreover, if for all nontrivial fully $\mathcal{F}$-centralised subgroups Q of P the bimodule M_Q induces a Morita equivalence between $A(\Delta Q)$ and $B(\Delta Q)$, then M and its dual M^* induce a stable equivalence of Morita type between A and B.

For V the trivial $\mathcal{O}P$-module, variations of the above result have been noted by many authors. For principal blocks this was first pointed out by Alperin. A version for finite groups with the same local structure appears in Broué [7, 6.3], and the above theorem with V trivial is equivalent to [15, Theorem 3.1]. The first class of examples for this situation with potentially nontrivial V goes back to work of Puig [23]: it is shown in [23, 6.8] that a block with an abelian defect group P and a Frobenius inertial quotient is stably equivalent to its Brauer correspondent, using the fact that the blocks of centralisers of nontrivial subgroups of P are nilpotent, hence Morita equivalent to the defect group algebra via a Morita equivalence with endopermutation source. The above theorem is used in the proof of Biland [3, Theorem 4.1] or [5, Theorem 1]. For convenience, we reformulate this at the block algebra level.

Theorem 1.2. *Let G, H be finite groups, and let b, c be blocks of $\mathcal{O}G$, $\mathcal{O}H$, respectively, having a common defect group P. Let $i \in (\mathcal{O}Gb)^{\Delta P}$ and $j \in (\mathcal{O}Hc)^{\Delta P}$ be almost source idempotents. For any subgroup Q of P denote by e_Q and f_Q the unique blocks of $kC_G(Q)$ and $kC_H(Q)$, respectively, satisfying $\mathrm{Br}_{\Delta Q}(i)e_Q \neq 0$ and $\mathrm{Br}_{\Delta Q}(j)f_Q \neq 0$. Denote by $\hat{e}_Q$ and $\hat{f}_Q$ the unique blocks of $\mathcal{O}C_G(Q)$ and $\mathcal{O}C_H(Q)$ lifting e_Q and f_Q, respectively.*