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A decomposition rule for certain tensor product representations of the symmetric groups

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In this paper, we give a combinatorial rule to calculate the decomposition of the tensor product (Kronecker product) of two irreducible complex representations of the symmetric group $\mathfrak{S}_n$, when one of the representations corresponds to a hook $(n-m, 1^m)$.

1. Introduction

Let G be a group such that each of its finite-dimensional complex representations is completely reducible. One of the basic problems of the representation theory of G is to give a decomposition rule

$$L(\lambda) \otimes L(\nu) \cong \bigoplus_{\mu \in P} m^{\mu}_{\lambda,\nu} L(\mu)$$

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of the tensor product of two irreducible representations of G, where $\{L(\lambda) \mid \lambda \in P\}$ denotes a set of complete representatives of irreducible finite-dimensional representations of G. When G is the general linear group $GL(n, \mathbb{C})$, the multiplicity $m^{\mu}_{\lambda,\nu}$ is known as the famous Littlewood–Richardson coefficient $LR^{\mu}_{\lambda,\nu}$, which equals the number of certain combinatorial objects (see e.g. [4]). When G is an arbitrary complex simple Lie group (or rather, an arbitrary symmetrizable Kac–Moody algebra), inspired by Kashiwara's theory of crystals and works of Lakshmibai and Seshadri, Littelmann [9] gave combinatorial objects whose number equals $m^{\mu}_{\lambda,\nu}$.

Let G be the symmetric group $\mathfrak{S}_n$ of n letters. As usual, we use the set $\mathcal{P}_n$ of partitions of n as the set P of labels of irreducible representations of G. In despite of the long research history (see e.g. Murnaghan [10] for an early work), much less is known about $m^{\mu}_{\lambda,\nu}$ for $\mathfrak{S}_n$-representations, comparing with the Lie theoretic case. Lascoux [7], Garsia and Remmel [5], Remmel [11], Remmel and Whitehead [12] and Rosas [14] gave descriptions of $m^{\mu}_{\lambda,\nu}$, when λ and ν are either two-row partitions or hook partitions. Recently, Ballantine and Orellana [1] gave a combinatorial rule for $m^{\mu}_{\lambda,\nu}$ in the case where λ is a two-row partition $(n-p, p)$ and ν is not a partition inside the $2(p-1) \times 2(p-1)$ square.

In this paper, we give a combinatorial rule to calculate the number $m^{\mu}_{\lambda,\nu}$ for $\mathfrak{S}_n$-representations, when the partition ν is a hook $(n-m, 1^m)$. More precisely, we construct a set $\mathrm{PH}_m(\lambda, \mu)$ in a combinatorial manner, which satisfies

$$L(\lambda) \otimes L(n-m, 1^m) \cong \bigoplus_{\mu \in \mathcal{P}_n} |\mathrm{PH}_m(\lambda, \mu)| \cdot L(\mu)$$

for each $\lambda, \mu \in \mathcal{P}_n$ and $0 \le m < n$.

Instead of dealing with the hook representation $L(n-m, 1^m)$ directly, we consider a slightly bigger representation $\Lambda_m(\mathbb{C}^n)$, the m-th exterior power of the defining representation of $\mathfrak{S}_n$. By considering a certain permutation representation $\mathbb{C}\Omega_m$, we show that the multiplicity of $L(\mu)$ in $L(\lambda) \otimes \Lambda_m(\mathbb{C}^n)$ is equal to $w_m := \sum_{\zeta \in \mathcal{P}_{n-m}} \sum_{\xi \in \mathcal{P}_m} LR^{\lambda}_{\zeta,\xi} \cdot LR^{\mu}_{\zeta,\xi^{\mathsf{t}}}$, where ξ^{t} denotes the transpose of ξ. Although we could not find the result in the literature, we suspect that it was known for experts, since the techniques to be used to prove are rather standard.

To give a decomposition of $L(\lambda) \otimes L(n-m, 1^m)$, we use a set $\mathrm{PW}_m(\lambda, \mu)$ such that $|\mathrm{PW}_m(\lambda, \mu)| = w_m$ and that each of its elements is a Zelevinsky's *picture* [16]. A picture is a bijective map between skew Young diagrams, which satisfies certain order-theoretic conditions, and it is identified with a tableau on a skew Young diagram satisfying some conditions. Using a variant of Zelevinsky's insertion algorithm for pictures [16], we construct the set $\mathrm{PH}_m(\lambda, \mu)$ as a subset of $\mathrm{PW}_m(\lambda, \mu)$.

2. Preliminaries

2.1. Partitions and diagrams

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ be a sequence of integers. We say that λ is a *partition* of n if $\lambda_1 \ge \dots \ge \lambda_l > 0$ and $|\lambda| := \sum_i \lambda_i = n$. For a partition $\lambda = (\lambda_1, \dots, \lambda_l)$, we define