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On the index of a free abelian subgroup in the group of central units of an integral group ring

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ABSTRACT

Let $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ denote the group of central units in the integral group ring $\mathbb{Z}[G]$ of a finite group G. A bound on the index of the subgroup generated by a virtual basis in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ is computed for a class of strongly monomial groups. The result is illustrated with application to the groups of order p^n, p prime, $n \leq 4$. The rank of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ and the Wedderburn decomposition of the rational group algebra of these p-groups have also been obtained.

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1. Introduction

Let $\mathcal{U}(\mathbb{Z}[G])$ denote the unit group of the integral group ring $\mathbb{Z}[G]$ of a finite group G. The center of $\mathcal{U}(\mathbb{Z}[G])$ is denoted by $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$. It is well known that $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G])) = \pm\mathcal{Z}(G) \times A$, where A is a free abelian subgroup of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ of finite rank. In order to study $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$, a multiplicatively independent subset of such a subgroup A, i.e., a $\mathbb{Z}$-basis for such a free $\mathbb{Z}$-module A, is of importance, and is known only for a few groups ([1,2,7,18], see also [20], Examples 8.3.11 and 8.3.12). However, other papers deal with determining a virtual basis of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$, i.e., a multiplicatively independent subset of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ which generates a subgroup of finite index in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ (see e.g. [6,9–17]).

Analogously to well known cyclotomic units in cyclotomic fields, Bass [4] constructed units, so called *Bass cyclic units*, which generate a subgroup of finite index in $\mathcal{U}(\mathbb{Z}[G])$, when G is cyclic. A virtual basis consisting of certain Bass cyclic units was also given by Bass. Generalizing the notion of Bass cyclic units, Jespers et al. [13] defined *generalized Bass units* and have shown that the group generated by these units contains a subgroup of finite index in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ for an arbitrary strongly monomial group G. Recently, for a class of groups properly contained in finite strongly monomial groups, Jespers et al. [15] provided a subset, denoted by $\mathcal{B}(G)$ (say), of the group generated by generalized Bass units, which forms a virtual basis of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$.

In this paper, we determine a bound on the index of the subgroup generated by $\mathcal{B}(G)$ in $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ for the same class of groups as considered in [15] (Theorem 2). Our result is based on the ideas contained in [15] and Kummer's work (see [23], Theorem 8.2) on the index of cyclotomic units. Further in [15], Jespers et al. have provided the rank of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ in terms of strong Shoda pairs of G, when G is a strongly monomial group. In Section 4, we compute a complete and irredundant set of strong Shoda pairs of the non abelian groups of order p^n, p prime, $n \leq 4$, and provide, in terms of p, the rank of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G]))$ of these p-groups along with the Wedderburn decomposition of their rational group algebras. We also illustrate Theorem 2 for the non abelian groups of order 16 and those of order p^3, $p \leq 5$. It may be mentioned that for a given group G, the calculation of the bound on the index given by Theorem 2 requires the values $n_{H,K}$ corresponding to the strong Shoda pairs (H, K) of G, the computation of which is not always obvious.

2. Notation and preliminaries

We begin by fixing some notation.

G	a finite group
$\|g\|$	the order of the element g in G
g^t	$t^{-1}gt,\ g, t \in G$
$\langle X \rangle$	the subgroup generated by the subset X of G
$\|X\|$	the cardinality of the set X
$K \leq G$	K is a subgroup of G