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Noncommutative Tsen's theorem in dimension one

A. Nyman

Department of Mathematics, 516 High St, Western Washington University, Bellingham, WA 98225-9063, United States

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ABSTRACT

Let k be a field. In this paper, we find necessary and sufficient conditions for a noncommutative curve of genus zero over k to be a noncommutative $\mathbb{P}^1$-bundle. This result can be considered a noncommutative, one-dimensional version of Tsen's theorem. By specializing this theorem, we show that every arithmetic noncommutative projective line is a noncommutative curve, and conversely we characterize exactly those noncommutative curves of genus zero which are arithmetic. We then use this characterization, together with results from [9], to address some problems posed in [4].

1. Introduction

Throughout this paper, k will denote a field. In [4], the concept of a noncommutative curve of genus zero is defined as a small k-linear abelian category H such that

- each object of H is noetherian,
- all morphism and extension spaces in H are finite dimensional over k,
- H admits an Auslander–Reiten translation, i.e. an autoequivalence τ such that Serre duality $\mathrm{Ext}^1_{\mathsf{H}}(\mathcal{E},\mathcal{F}) \cong D\,\mathrm{Hom}_{\mathsf{H}}(\mathcal{F},\tau\mathcal{E})$ holds, where $D(-)$ denotes the k-dual,

E-mail address: adam.nyman@wwu.edu.

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- H has an object of infinite length, and
- H has a tilting object.

One motivation for the definition is that if C is a smooth projective curve of genus zero over k, then the category of coherent sheaves over C satisfies these properties. Furthermore, Kussin calls the category H *homogeneous* if

- for all simple objects $\mathcal{S}$ in H, $\mathrm{Ext}^1_{\mathsf{H}}(\mathcal{S}, \mathcal{S}) \neq 0$.

If H is not homogeneous (e.g. if H is a weighted projective line) then H is birationally equivalent to a homogeneous noncommutative curve of genus zero [4, p. 2]. Therefore, from the perspective of noncommutative birational geometry, the homogeneous curves play a crucial role.

If H is a homogeneous noncommutative curve of genus zero and $\mathcal{L}$ is a line bundle on H, then there exists an indecomposable bundle $\overline{\mathcal{L}}$ and an irreducible morphism $\mathcal{L} \longrightarrow \overline{\mathcal{L}}$ coming from an AR sequence starting at $\mathcal{L}$. Kussin calls the bimodule $M := {}_{\mathrm{End}(\overline{\mathcal{L}})}\mathrm{Hom}_{\mathsf{H}}(\mathcal{L}, \overline{\mathcal{L}})_{\mathrm{End}(\mathcal{L})}$ the *underlying bimodule* of H. It turns out that the only possibilities for the left-right dimensions of M are $(1, 4)$ and $(2, 2)$.

On the other hand, in [14], M. van den Bergh introduces the notion of a noncommutative $\mathbb{P}^1$-bundle over a pair of commutative schemes X, Y. In particular, if K and L are finite extensions of k and N is a k-central $K - L$-bimodule of finite dimension as both a K-module and an L-module, then one can form the $\mathbb{Z}$-algebra $\mathbb{S}^{n.c.}(N)$, the noncommutative symmetric algebra of N (see Section 2 for details). The *noncommutative* $\mathbb{P}^1$*-bundle generated by* N, $\mathbb{P}^{n.c.}(N)$, is defined to be the quotient of the category of graded right $\mathbb{S}^{n.c.}(N)$-modules modulo the full subcategory of direct limits of right bounded modules. It is natural to ask whether a homogeneous noncommutative curve of genus zero is a noncommutative $\mathbb{P}^1$-bundle generated by M, at least under the necessary condition that $\mathrm{End}(\mathcal{L})$ and $\mathrm{End}(\overline{\mathcal{L}})$ are commutative. Our main result is that this is the case. Before we state it precisely, we need to introduce some notation. If C is a noetherian category, then there exists a unique locally noetherian category $\tilde{\mathsf{C}}$ whose full subcategory of noetherian objects is C [13, Theorem 2.4]. Furthermore, if C and D are k-linear categories and there exists a k-linear equivalence $\mathsf{C} \longrightarrow \mathsf{D}$, we write $\mathsf{C} \equiv \mathsf{D}$.

Our main theorem is the following (Theorem 3.10):

Theorem 1.1. *If* H *is a homogeneous noncommutative curve of genus zero with underlying bimodule* M *such that* $\mathrm{End}(\mathcal{L})$ *and* $\mathrm{End}(\overline{\mathcal{L}})$ *are commutative, then*

$$\tilde{\mathsf{H}} \equiv \mathbb{P}^{n.c.}(M).$$

Conversely, if K *and* L *are finite extensions of* k *and* N *is a* k*-central* $K - L$*-bimodule of left-right dimension* $(2, 2)$ *or* $(1, 4)$*, then* $\mathbb{P}^{n.c.}(N)$ *is a noncommutative curve of genus zero with underlying bimodule* N*.*