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Graded algebras with polynomial growth of their codimensions

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ABSTRACT

Let A be an algebra over a field of characteristic 0 and assume A is graded by a finite group G. We study combinatorial and asymptotic properties of the G-graded polynomial identities of A provided A is of polynomial growth of the sequence of its graded codimensions. Roughly speaking this means that the ideal of graded identities is "very large". We relate the polynomial growth of the codimensions to the module structure of the multilinear elements in the relatively free G-graded algebra in the variety generated by A. We describe the irreducible modules that can appear in the decomposition, we show that their multiplicities are eventually constant depending on the shape obtained by the corresponding multipartition after removing its first row. We relate, moreover, the polynomial growth to the colengths. Finally we describe in detail the algebras whose graded codimensions are of linear growth.

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Introduction

All algebras we consider are associative, not necessarily unitary, and over a fixed field F of characteristic 0. Let A be an associative algebra satisfying a polynomial identity (also called a PI algebra), and let G be a finite group. Assume that A is G-graded, then A satisfies non-trivial G-graded polynomial identities. Denote by P_n the vector space of all multilinear polynomials of degree n in the variables $x_1, \ldots, x_n$ in the free associative algebra $F\langle X\rangle$ freely generated over F by $X = \{x_1, x_2, \ldots\}$. It is well known that in order to study the polynomial identities of A one may consider only the multilinear ones (as long as the characteristic of the base field equals 0). If A is an algebra and $\mathrm{Id}(A)$ is its T-ideal, that is the ideal of its polynomial identities in $F\langle X\rangle$ then $\mathrm{Id}(A)$ is generated as a T-ideal by the elements in $\mathrm{Id}(A) \cap P_n$ for $n \geq 1$. The vector space P_n is a left module over the symmetric group in a natural way, and it is isomorphic to the left regular S_n-module FS_n, and moreover $P_n \cap \mathrm{Id}(A)$ is its submodule. It is more convenient to consider the factor module $P_n(A) = P_n/(P_n \cap \mathrm{Id}(A))$ instead of $P_n \cap \mathrm{Id}(A)$. Following this line one applies the theory of representations of the symmetric group to the study of PI algebras, and in an equivalent form, the representations of the general linear group. Hence it is important to know the decomposition of $P_n(A)$ into irreducible modules, its character, the generators of the irreducible modules and so on. One of the most important numerical invariants of a PI algebra is its *codimension sequence* $c_n(A) = \dim P_n(A)$. Despite its importance the exact computation of the codimensions of an algebra is extremely difficult, and it has been done for very few algebras. That is why one is led to study the asymptotic behaviour of the codimensions. A celebrated theorem of Regev asserts that if A is a PI algebra satisfying an identity of degree d then $c_n(A) \leq (d-1)^{2n}$. Thus the growth of the sequence $c_n(A)$ cannot be very "fast": $\dim P_n = n!$. In the late nineties Giambruno and Zaicev (see for example their monograph [17]) proved that if A is a PI algebra then the limit $\lim_{n\to\infty}(c_n(A)^{1/n})$ exists and is always a non-negative integer called the *exponent* (or PI exponent) of the algebra A, denoted by $\exp(A)$, thus answering in the affirmative a conjecture of Amitsur. The PI exponent of A can be explicitly computed; it is closely related to the structure of A and equals the dimension of certain semisimple algebra related to A.

One may thus classify the PI algebras according to their exponents. Of special interest are the PI algebras with "slow" codimension growth. It is well known that $\exp(A) \leq 1$ if and only if $c_n(A)$ is polynomially bounded. Various descriptions of such algebras were given, the interested reader might want to consult the monographs [17, Chapter 7] for further information about this topic. We recall that a theorem of Kemer [20,21] states that the following conditions are equivalent for the PI algebra A.

(1) The codimension sequence $c_n(A)$ is polynomially bounded.
(2) $\exp(A) \leq 1$.