Journal of Algebra 434 (2015) 227–248

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On Sidki's presentation for orthogonal groups

Justin M^c^Inroy [a,*,1], Sergey Shpectorov [b,2]

[a] Heilbronn Institute for Mathematical Research, School of Mathematics, University of Bristol, University Walk, Bristol, BS8 1TW, UK
[b] School of Mathematics, University of Birmingham, Edgbaston, Birmingham, B15 2TT, UK

ARTICLE INFO

Article history:
Received 16 November 2014
Available online 17 April 2015
Communicated by Martin Liebeck

Keywords:
Orthogonal groups
Presentations
Matrix groups over rings
Clifford algebra

ABSTRACT

We study presentations, defined by Sidki, resulting in groups $y(m,n)$ that are conjectured to be finite orthogonal groups of dimension $m+1$ in characteristic two. This conjecture, if true, shows an interesting pattern, possibly connected with Bott periodicity. It would also give new presentations for a large family of finite orthogonal groups in characteristic two, with no generator having the same order as the cyclic group of the field.

We generalise the presentation to an infinite version $y(m)$ and explicitly relate this to previous work done by Sidki. The original groups $y(m,n)$ can be found as quotients over congruence subgroups of $y(m)$. We give two representations of our group $y(m)$. One into an orthogonal group of dimension $m+1$ and the other, using Clifford algebras, into the corresponding pin group, both defined over a ring in characteristic two. Hence, this gives two different actions of the group. Sidki's homomorphism into $SL_{2^{m-2}}(R)$ is recovered and extended as an action on a submodule of the Clifford algebra.

* Corresponding author.
E-mail addresses: justin.mcinroy@bristol.ac.uk (J. M^c^Inroy), S.Shpectorov@bham.ac.uk (S. Shpectorov).
[1] Fax: +44 117 331 5264.
[2] Fax: +44 121 414 3389.

http://dx.doi.org/10.1016/j.jalgebra.2015.02.031

Table 1
Some small cases.

$Y(3,5) \cong SL_2(16) \cong \Omega^-(4,4)$	$Y(3,7) \cong \Omega^+(4,8)$
$Y(4,5) \cong \Omega(5,4)$	$Y(4,7) \cong \Omega(5,8)$
$Y(5,5) \cong \Omega^-(6,4)$	$Y(5,7) \cong \Omega^+(6,8)$
$Y(6,5) \cong 4^6 : \Omega^-(6,4)$	$Y(6,7) \cong 8^6 : \Omega^+(6,8)$
$Y(7,5) \cong \Omega^-(8,4)$	
$Y(8,5) \cong \Omega(9,4)$	$Y(3,11) \cong \Omega^-(4,32)$
$Y(9,5) \cong \Omega^-(10,4)$	$Y(4,11) \cong \Omega(5,32)$
$Y(10,5) \cong 4^{10} : \Omega^-(10,4)$	$Y(5,11) \cong \Omega^-(6,32)$

1. Introduction

The following is a well-known presentation of the alternating group A_{m+2} which was given by Carmichael in 1923:

$$A_{m+2} = \langle a_1, \dots, a_m \mid a_i^3 = 1, (a_i a_j)^2 = 1, \text{ for all } i \neq j \rangle.$$

In 1982, Sidki generalised this to the following:

$$Y(m,n) := \langle a_1, \dots, a_m \mid a_i^n = 1, (a_i^k a_j^k)^2 = 1, \text{ for all } i \neq j, 1 \leq k \leq n-1 \rangle$$

and he conjectured that these too are all finite groups.

It is clear that $Y(m,3)$ is just Carmichael's presentation for the alternating group. In [2,3], Sidki identifies the groups $Y(2,n)$, $Y(3,n)$ (when n is odd), $Y(m,2)$ and $Y(m,4)$ and hence shows that they are finite. In general, however, the question of finiteness remains open.

As well as identifying the groups when either m, or n is small, Sidki gives some general results. He shows that $Y(m,n)$ is perfect provided $m > 2$ and n is odd, and that if $n | n'$, then $Y(m,n)$ is a quotient group of $Y(m,n')$ [2, Theorem A]. The second of these results allows a reduction to the cases where n is a prime power. When $n = 2^r$, it is conjectured that $Y(m, 2^r)$ is a 2-group. If n is odd, however, Sidki shows that $Y(m,n)$ is isomorphic to the following group:

$$\begin{aligned} y(m,n) := \langle a, S_m \mid \; & a^n = 1, [s_1, s_1^{a^k}] = 1, 1 \leq k \leq n-1 \\ & s_1^{1+a+\dots+a^{n-1}} = 1 \text{ and } a^{s_i} = a^{-1}, 2 \leq i \leq m-1 \rangle, \end{aligned}$$

where S_m denotes the symmetric group and $s_i = (i, i+1)$.

In addition to the results above, a few small cases were resolved by J. Neubuser, W. Felsch and E. O'Brian by direct calculations using the Todd–Coxeter algorithm (see Table 1). These calculations suggest that, for odd n, $y(m,n)$ are orthogonal groups in characteristic two of dimension $m+1$ for some suitable quadratic form. Note that when there is a normal subgroup, namely when $m = 2 \bmod 4$, this agrees with our assessment and just indicates that the form has a non-trivial radical.