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Formal geometry for noncommutative manifolds

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ABSTRACT

This paper develops the tools of formal algebraic geometry in the setting of noncommutative manifolds, roughly ringed spaces locally modeled on the free associative algebra. We define a notion of noncommutative coordinate system, which is a principal bundle for an appropriate group of local coordinate changes. These bundles are shown to carry a natural flat connection with properties analogous to the classical Gelfand–Kazhdan structure.

Every noncommutative manifold has an underlying smooth variety given by abelianization. A basic question is existence and uniqueness of noncommutative thickenings of a smooth variety, i.e., finding noncommutative manifolds abelianizing to a given smooth variety. We obtain new results in this direction by showing that noncommutative coordinate systems always arise as reductions of structure group of the commutative bundle of coordinate systems on the underlying smooth variety; this also explains a relationship between $\mathcal{D}$-modules on the commutative variety and sheaves of modules for the noncommutative structure sheaf.

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1. Introduction

Throughout X is a smooth variety over $\mathbb{C}$ of dimension n.

Definition 1.1. Let A be an associative $\mathbb{C}$-algebra. Define the lower central series filtration as follows. Set $L_1(A) = A$, and $L_k(A) = [A, L_{k-1}(A)]$. Then the *lower central series ideals* are

$$M_k(A) = AL_k(A)A = AL_k(A).$$

We say that A is *NC-complete* if A is complete for the filtration $M_k(A)$. The *NC-completion* of an algebra is its completion with respect to the filtration M_k.

The remarkable paper of Kapranov [12] puts forward a framework of noncommutative geometry in which the local objects are NC-complete algebras with a smoothness property.

Definition 1.2. Let A be an NC-complete $\mathbb{C}$-algebra, $\pi : A \to A_{ab}$ the abelianization map, and $x \in \operatorname{Spec} A_{ab}$. Denote by $S = \pi^{-1}(\overline{S}) \subset A$ the multiplicative subset corresponding to a multiplicative subset $\overline{S} \subset A_{ab}$.

1 The *stalk of A at x* is the direct limit of all localizations $A[S^{-1}]$, where $\overline{S} \subset A_{ab}$ runs over multiplicative subsets of functions not vanishing at x.
2 An NC-complete algebra A is *NC-smooth of dimension n* if all of its completed stalks are isomorphic to $\widehat{A}_n = k\langle\langle x_1, \ldots, x_n\rangle\rangle$.