Journal of Algebra 436 (2015) 1–16

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On σ-subnormal and σ-permutable subgroups of finite groups

Alexander N. Skiba [1]

Department of Mathematics, Francisk Skorina Gomel State University, Gomel 246019, Belarus

ARTICLE INFO

Article history:
Received 17 January 2015
Available online 26 April 2015
Communicated by E.I. Khukhro

MSC:
20D10
20D15
20D30

Keywords:
Finite group
σ-Group
A complete Hall set of type σ
σ-Subnormal subgroup
σ-Permutable subgroup

ABSTRACT

Let $\sigma = \{\sigma_i | i \in I\}$ be some partition of the set $\mathbb{P}$ of all primes, that is, $\mathbb{P} = \cup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. Let G be a finite group. We say that G is: *σ-primary* if G is a σ_i-group for some $i \in I$; *σ-soluble* if every chief factor of G is σ-primary. We say that a set $\mathcal{H} = \{H_1, \ldots, H_t\}$ of Hall subgroups of G, where H_i is σ-primary ($i = 1, \ldots, t$), is a *complete Hall set of type σ* of G if $(|H_i|, |H_j|) = 1$ for all $i \neq j$ and $\pi(G) = \pi(H_1) \cup \cdots \cup \pi(H_t)$. We say that a subgroup A of G is: *σ-subnormal* in G if there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_n = G$ such that either A_{i-1} is normal in A_i or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \ldots, t$; *σ-permutable* in G if G has a complete Hall set $\mathcal{H}$ of type σ such that A is $\mathcal{H}^G$-permutable in G, that is, $AH^x = H^xA$ for all $x \in G$ and all $H \in \mathcal{H}$. We study the relationship between the σ-subnormal and σ-permutable subgroups of G. In particular, we prove that every σ-permutable subgroup of G is σ-subnormal, and we classify finite σ-soluble groups in which every σ-subnormal subgroup is σ-permutable.

E-mail address: alexander.skiba49@gmail.com.

[1] Research of the author was supported by State Program of Fundamental Researches of Republic Belarus (Grant 20112850).

http://dx.doi.org/10.1016/j.jalgebra.2015.04.010
0021-8693/

1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. If n is an integer, then the symbol $\pi(n)$ denotes the set of all primes dividing $|n|$; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G.

In what follows, $\sigma = \{\sigma_i | i \in I\}$ is some partition of $\mathbb{P}$, that is, $\mathbb{P} = \cup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$.

We put $\sigma(n) = \{\sigma_i \cap \pi(n) \mid i \in I, \sigma_i \cap \pi(n) \neq \emptyset\}$, $\sigma(G) = \sigma(|G|)$ and we say that G is *σ-primary* if either $G = 1$ or $|\sigma(G)| = 1$.

A set $\mathcal{S}$ of Sylow subgroups of G is called a *complete set of Sylow subgroups* of G if $\mathcal{S}$ contains exact one Sylow p-subgroup of G for every prime p dividing $|G|$. By analogy with it, we say that a set $\mathcal{H} = \{H_1, \ldots, H_t\}$ of Hall subgroups of G, where H_i is σ-primary ($i = 1, \ldots, t$), is a *complete Hall set of G of type σ* if $(|H_i|, |H_j|) = 1$ for all $i \neq j$ and $\pi(G) = \pi(H_1) \cup \cdots \cup \pi(H_t)$. Following [1], the group G is a *σ-group* if it possesses a complete Hall set of type σ.

Let $\mathcal{L}$ be some non-empty set of subgroups of G. Then a subgroup A of G is called *$\mathcal{L}$-permutable* if $AH = HA$ for all $H \in \mathcal{L}$; *$\mathcal{L}^G$-permutable* if $AH^x = H^xA$ for all $H \in \mathcal{L}$ and all $x \in G$. If A is $\mathcal{L}^G$-permutable, where $\mathcal{L}$ is a complete set of Sylow subgroups of G, then A is called *S-quasinormal* or *S-permutable* in G. By analogy with it, we say that a subgroup A of G is *σ-permutable* in G if G has a complete Hall set $\mathcal{H}$ of type σ such that A is $\mathcal{H}^G$-permutable.

The S-permutable subgroups possess a series of interesting properties and they are closely related to subnormal subgroups. For instance, if H is an S-permutable subgroup of G, then H is subnormal in G (Kegel [2]) and the quotient H/H_G is nilpotent (Deskins [3]) and so, in fact, H^G/H_G is nilpotent. Moreover, H/H_G is nilpotent also in the case when H is subnormal and permutes with all members of some complete set of Sylow subgroups of G [4]. The S-permutable subgroups of G form a sublattice of the lattice of all subnormal subgroups of G (Kegel [2]). The description of PST-groups, that are groups, in which every subnormal subgroup is S-permutable, was first obtained by Agrawal [5], for the soluble case, and by Robinson in [6], for the general case. In the further publications, authors (see, for example, the recent papers [7–16]) have found out and described many other interesting characterizations of soluble PST-groups.

In fact, the following concept is a modification of the main concept in [17].

Definition 1.1. We say that a subgroup A of G is *σ-subnormal* in G if there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_n = G$ such that either A_{i-1} is normal in A_i or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \ldots, n$.

Before continuing, let's consider the following elementary example.

Example 1.2. Let p, q, r, t be different primes, where q and r divide $p - 1$. Let A be a non-abelian group of order pr, T a group of order t and $P \rtimes Q$ a non-abelian group of