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Real reductive Cayley groups of rank 1 and 2 [☆]

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ABSTRACT

A linear algebraic group G over a field K is called a Cayley K-group if it admits a Cayley map, i.e., a G-equivariant K-birational isomorphism between the group variety G and its Lie algebra. We classify real reductive algebraic groups of absolute rank 1 and 2 that are Cayley $\mathbb{R}$-groups.

1. Introduction

Let G be a connected linear algebraic group defined over a field K, and let $\mathrm{Lie}(G)$ denote its Lie algebra. The following definitions are due to Lemire, Popov and Reichstein [12]:

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Definitions 1.1. (See [12].) A *Cayley map for G* is a K-birational isomorphism $G \overset{\simeq}{\dashrightarrow} \mathrm{Lie}(G)$ which is G-equivariant with respect to the action of G on itself by conjugation and the action of G on $\mathrm{Lie}(G)$ via the adjoint representation. A linear algebraic K-group is called a *Cayley group* if it admits a Cayley map. A linear algebraic K-group is called a *stably Cayley group* if $G \times_K (\mathbb{G}_{m,K})^r$ is Cayley for some $r \geq 0$, where $\mathbb{G}_{m,K}$ denotes the multiplicative group.

Lemire, Popov and Reichstein [12] classified Cayley and stably Cayley simple groups over an algebraically closed field k of characteristic 0. Borovoi, Kunyavskiĭ, Lemire and Reichstein [2] classified stably Cayley simple K-groups, and later Borovoi and Kunyavskiĭ [3] classified stably Cayley semisimple K-groups, over an arbitrary field K of characteristic 0. Clearly any Cayley K-group is stably Cayley. In the opposite direction, some of the stably Cayley K-groups are known to be Cayley, see [12, Examples 1.9, 1.11 and 1.16]. For other stably Cayley K-groups, it is a difficult problem to determine whether they are Cayley or not. By [2, Lemma 5.4(c)] the answer to the question whether a K-group is Cayley depends only on the equivalence class of G up to inner twisting.

By [2, Corollary 7.1] all the reductive K-groups of rank ≤ 2 over a field K of characteristic 0 are stably Cayley (by the rank we always mean the *absolute* rank). We would like to know, which of those stably Cayley K-groups of rank ≤ 2 are Cayley.

The case of a simple group of type $\mathbf{G}_2$ was settled in [12, §9.2] and Iskovskikh's papers [9,10]. Namely, a simple group of type $\mathbf{G}_2$ over an algebraically closed field k of characteristic 0 is not Cayley. Hence no simple K-group of type $\mathbf{G}_2$ over a field K of characteristic 0 is Cayley.

Popov [15] proved in 1975 that, contrary to what was expected (cf. [13, Remarque, p. 14]), the group $\mathbf{SL}_3$ over an *algebraically closed field* k of characteristic 0 is Cayley; see [12, Appendix] for Popov's original proof, and [12, §9.1] for an alternative proof.

Here we are interested in $\mathbb{R}$-groups, where $\mathbb{R}$ denotes the field of real numbers. If G is an inner form of a split reductive $\mathbb{R}$-group, and $G_{\mathbb{C}} := G \times_{\mathbb{R}} \mathbb{C}$ is *stably Cayley* over $\mathbb{C}$, then by [2, Remark 1.8] G is *stably Cayley* over $\mathbb{R}$. Similarly, since $\mathbf{SL}_{3,\mathbb{C}}$ is *Cayley* over $\mathbb{C}$ by Popov's theorem, one might expect that the split $\mathbb{R}$-group $\mathbf{SL}_{3,\mathbb{R}}$ is *Cayley* over $\mathbb{R}$. However, this turns out to be false, see Theorem 8.1 of Appendix A. On the other hand, the outer form $\mathbf{SU}_3$ of the split group $\mathbf{SL}_{3,\mathbb{R}}$ is Cayley, see Theorem 7.1 of Appendix A and Corollary 4.4.

We recall the classification of reductive K-groups of rank ≤ 2. A reductive K-group of rank 1 is either a K-torus or a simple K-group of type $\mathbf{A}_1$. A reductive K-group of rank 2 is either not semisimple, or semisimple of type $\mathbf{A}_1 \times \mathbf{A}_1$, or simple of one of the types $\mathbf{A}_2$, $\mathbf{B}_2 = \mathbf{C}_2$, or $\mathbf{G}_2$. If a reductive K-group of rank 2 is not semisimple, then either it is a K-torus or it is isogenous to the product of a one-dimensional K-torus and a simple K-group of type $\mathbf{A}_1$.

We recall the classification of *real* simple groups of type $\mathbf{A}_2$. Such an $\mathbb{R}$-group is isomorphic to one of the groups $\mathbf{SL}_{3,\mathbb{R}}$, $\mathbf{PGL}_{3,\mathbb{R}}$, $\mathbf{SU}_3$, $\mathbf{PGU}_3$, $\mathbf{SU}(2,1)$, or $\mathbf{PGU}(2,1)$. Here, following the Book of Involutions [11, §23], we write $\mathbf{PGU}_n$ rather than $\mathbf{PSU}_n$ for