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Algebraic connections on projective modules with prescribed curvature

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ABSTRACT

In this paper we generalize some results on universal enveloping algebras of Lie algebras to Lie–Rinehart algebras and twisted universal enveloping algebras of Lie–Rinehart algebras. We construct for any Lie–Rinehart algebra L and any 2-cocycle f in $\mathrm{Z}^2(L, A)$ the universal enveloping algebra $U(f)$ of type f. When L is projective as left A-module we prove a PBW-Theorem for $U(f)$ generalizing classical PBW-Theorems. We then use this construction to give explicit constructions of a class of finitely generated projective A-modules with no flat algebraic connections. One application of this is that for any Lie–Rinehart algebra L which is projective as left A-module and any cohomology class c in $\mathrm{H}^2(L, A)$ there is a finite rank projective A-module E with $c_1(E) = c$. Another application is to construct for any Lie–Rinehart algebra L which is projective as left A-module a subring $Char(L)$ of $\mathrm{H}^*(L, A)$ – the characteristic ring of L. The ring $Char(L)$ ring is defined in terms of the cohomology group $\mathrm{H}^2(L, A)$ and has the property that it is a non-trivial subring of the image of the Chern character $Ch_{\mathbf{Q}} : \mathrm{K}(L)_{\mathbf{Q}} \to \mathrm{H}^*(L, A)$. We also give an explicit realization of the category of L-connections as a category of modules on an associative algebra $U^{ua}(L)$.

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Contents

1. Introduction

In the following paper we generalize classical notions on Lie algebras and universal enveloping algebras of Lie algebras (see [14] and [16]) to Lie–Rinehart algebras and universal enveloping algebras of Lie–Rinehart algebras. We construct for any Lie–Rinehart algebra L and any 2-cocycle f in $\mathrm{Z}^2(L, A)$, a generalized universal enveloping algebra $U(A, L, f)$. The algebra $U(A, L, f)$ is equipped with an increasing and decreasing filtration. When $f = 0$ we get Rinehart's universal enveloping algebra $U(A, L)$. We prove the following Poincaré–Birkhoff–Witt Theorem for $U(B, L, f)$ (see Theorem 3.7):

Theorem 1.1. *Let L be projective as left A-module. There is a canonical isomorphism of graded A-algebras*

$$\mathrm{Sym}^*_A(L) \cong Gr(U(A, L, f)),$$

where $Gr(U(A, L, f))$ is the associated graded algebra of $U(A, L, f)$ with respect to the ascending filtration.

As a consequence we get new examples of finitely generated projective modules with no flat algebraic connections. We also construct families of (mutually non-isomorphic) finitely generated projective modules of arbitrary high rank using filtrations in the algebra $U(A, L, f)$ (see Example 5.12).

Let $c \in \mathrm{H}^2(L, A)$ be any cohomology class. We use sub-quotients of the universal enveloping algebra $U(A, L, f)$ to prove the following theorem (see Theorem 5.3).

Theorem 1.2. *Let R contain a field of characteristic zero and let L be projective as left A-module. There exist for every pair of integers $k, i \geq 1$ a finitely generated projective A-module $V(k, i)$ with the following property:*

$$c_1(V(k, i)) = c \in \mathrm{H}^2(L, A).$$

Moreover $rk(V(k, i)) = \binom{l+k+i-1}{l} - \binom{l+k-1}{l}$.