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Spin structures on flat manifolds

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A B S T R A C T

We present an algorithmic approach to the problem of the existence of spin structures on flat manifolds. We apply our method in the cases of flat manifolds of dimensions 5 and 6.

1. Introduction

Let Γ be an n-dimensional crystallographic group, i.e. a discrete and cocompact subgroup of the group $E(n) = O(n) \ltimes \mathbb{R}^n$ of isometries of the Euclidean space $\mathbb{R}^n$. By the Bieberbach theorems (see [1–3]), Γ fits into short exact sequence

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \longrightarrow G \longrightarrow 1, \tag{1}$$

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where $\mathbb{Z}^n$ is a maximal abelian normal subgroup of Γ and G is a finite group, the so-called holonomy group of Γ. When in addition Γ is torsionfree, then Γ is called a Bieberbach group. In this case the orbit space $\mathbb{R}^n/\Gamma$ is a flat manifold, i.e. a closed connected Riemannian manifold with sectional curvature equal to zero.

The existence of a spin structure on a manifold X allows us to define on X a Dirac operator. Every oriented flat manifold of dimension less than or equal to 3 admits a spin structure. In dimension 4, 24 out of 27 flat manifolds have spin structures (see [16]). In this paper we present an algorithm to determine the existence of a spin structure on a flat manifold and present some facts concerning spin structures on flat manifolds of dimensions 5 and 6.

Section 2 recalls some basic definitions and introduces the necessary notations concerning Clifford algebras. The main goal of Section 3 is to present a more flexible form of a Pfäffle criterion of the existence of spin structures on flat manifolds. The key tool in looking for spin structures on a flat manifold is the restriction of its holonomy representation to the Sylow 2-subgroup of the holonomy group. In Section 4 we show that this restriction can be realized in a very convenient form and in Section 5 we show its usage in the criterion mentioned above. The algorithm for determining spin structures on flat manifolds is presented in Section 6 and is followed by an example of its usage for a 5-dimensional flat manifold. The last section presents some facts about spin structures for 5- and 6-dimensional manifolds.

2. Clifford algebras and spin groups

Definition 1. Let $n \in \mathbb{N}$. The *Clifford algebra* C_n is a real associative algebra with one, generated by elements $e_1, \dots, e_n$, which satisfy relations:

$$\forall_{1 \leq i < j \leq n}\ e_i^2 = -1 \text{ and } e_i e_j = -e_j e_i.$$

Remark 1. We have the following $\mathbb{R}$-algebras isomorphisms:

$$C_0 \cong \mathbb{R}, \quad C_1 \cong \mathbb{C}, \quad C_2 \cong \mathbb{H}.$$

Remark 2. We may view $\mathbb{R}^n := \operatorname{span}\{e_1, \dots, e_n\}$ as a vector subspace of C_n, for $n \in \mathbb{N}$.

Definition 2 *(Three involutions).* Let $n \in \mathbb{N}$. We have the following involutions of C_n:

- ${}^*\colon C_n \to C_n$, defined on the basis of (the vector space) C_n by

$$\forall_{1 \leq i_1 < i_2 < \ldots < i_k \leq n}\ (e_{i_1} \ldots e_{i_k})^* = e_{i_k} \ldots e_{i_1};$$

- ${}'\colon C_n \to C_n$, defined on the generators of (the algebra) C_n by

$$\forall_{1 \leq i \leq n} e_i' = -e_i.$$