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On endotrivial modules for Lie superalgebras

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A R T I C L E I N F O

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A B S T R A C T

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra over an algebraically closed field, k, of characteristic 0. An endotrivial $\mathfrak{g}$-module, M, is a $\mathfrak{g}$-supermodule such that $\operatorname{Hom}_k(M, M) \cong k_{ev} \oplus P$ as $\mathfrak{g}$-supermodules, where k_{ev} is the trivial module concentrated in degree $\bar{0}$ and P is a projective $\mathfrak{g}$-supermodule. In the stable module category, these modules form a group under the operation of the tensor product. We show that for an endotrivial module M, the syzygies $\Omega^n(M)$ are also endotrivial, and for certain Lie superalgebras of particular interest, we show that $\Omega^1(k_{ev})$ and the parity change functor actually generate the group of endotrivials. Additionally, for a broader class of Lie superalgebras, for a fixed n, we show that there are finitely many endotrivial modules of dimension n.

1. Introduction

The study of endotrivial modules began with Dade in 1978 when he defined endotrivial kG-modules for a finite group G in [16] and [17]. Endotrivial modules arose naturally in this context and play an important role in determining the simple modules for p-solvable groups. Dade showed that, for an abelian p-group G, endotrivial kG-modules have the form $\Omega^n(k) \oplus P$ for some projective module P, where $\Omega^n(k)$ is the nth syzygy (defined

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in Section 2) of the trivial module k. In general, in the stable module category, the endotrivial modules form an abelian group under the tensor product operation. It is known, via Puig in [21], that this group is finitely generated in the case of kG-modules and is completely classified for p-groups over a field of characteristic p by Carlson and Thévenaz in [13] and [14]. An important step in this classification is a technique where the modules in question are restricted to elementary abelian subgroups.

Carlson, Mazza, and Nakano have also computed the group of endotrivial modules for finite groups of Lie type (in the defining characteristic) in [9]. The same authors in [10] and Carlson, Hemmer, and Mazza in [8] give a classification of endotrivial modules for the case when G is either the symmetric or alternating group.

This class of modules has also been studied for modules over finite group schemes by Carlson and Nakano in [11]. The authors show that all endotrivial modules for a unipotent abelian group scheme have the form $\Omega^n(k) \oplus P$ in this case as well. For certain group schemes of this type, a classification is also given in the same paper (see Section 4). The same authors proved, in an extension of this paper, that given an arbitrary finite group scheme, for a fixed n, the number of isomorphism classes of endotrivial modules of dimension n is finite (see [12]), but it is not known whether the endotrivial group is finitely generated in this context.

We wish to extend the study of this class of modules to Lie superalgebra modules. First we must establish the correct notion of endotrivial module in this context. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra over an algebraically closed field, k, of characteristic 0. A $\mathfrak{g}$-supermodule, M, is called endotrivial if there is a supermodule isomorphism $\mathrm{Hom}_k(M, M) \cong k_{ev} \oplus P$ where k_{ev} is the trivial supermodule concentrated in degree $\bar{0}$ and P is a projective $\mathfrak{g}$-supermodule.

There are certain subalgebras, denoted $\mathfrak{e}$ and $\mathfrak{f}$, of special kinds of classical Lie superalgebras which are of interest. These subalgebras "detect" the cohomology of the Lie superalgebra $\mathfrak{g}$. By this, we mean that the cohomology for $\mathfrak{g}$ embeds into particular subrings of the cohomology for $\mathfrak{e}$ and $\mathfrak{f}$. These detecting subalgebras can be considered analogous to elementary abelian subgroups and are, therefore, of specific interest.

In this paper, we observe that the universal enveloping Lie superalgebra $U(\mathfrak{e})$ has a very similar structure to the group algebra kG when G is abelian, noncyclic of order 4 and $\operatorname{char} k = 2$ (although $U(\mathfrak{e})$ is not commutative). With this observation, we draw from the results of [7] to prove the base case in an inductive argument for the classification of the group of endotrivial $U(\mathfrak{e})$-supermodules. The inductive step uses techniques from [11] to complete the classification. For the other detecting subalgebra $\mathfrak{f}$, even though $U(\mathfrak{f})$ is not isomorphic to $U(\mathfrak{e})$, reductions are made to reduce this case to the same proof.

The main result is that for the detecting subalgebras $\mathfrak{e}$ and $\mathfrak{f}$, denoted generically as $\mathfrak{a}$, the group of endotrivial supermodules, $T(\mathfrak{a})$, is isomorphic to $\mathbb{Z}_2$ when the rank of $\mathfrak{a}$ is one and $\mathbb{Z} \times \mathbb{Z}_2$ when the rank is greater than or equal to two.

We also show that for a classical Lie superalgebra $\mathfrak{g}$ such that there are finitely many simple $\mathfrak{g}_{\bar{0}}$-modules of dimension $\leq n$, there are only finitely many endotrivial $\mathfrak{g}$-supermodules of a fixed dimension n. This is done by considering the variety of