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A note on locality of algebras

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ABSTRACT

In [15], the second author stated two conjectures characterizing algebraic commutative local algebras with unit over a field F. In this paper we settle Conjecture 1 from [15] in positive, and Conjecture 2 in positive when char $F = 0$ and in negative otherwise. We also discuss locality of arbitrary commutative algebras with unit in this context. As an application, we propose deterministic and randomized polynomial-time algorithms that test locality of finite-dimensional algebras over algebraic number fields.

1. Introduction

In commutative algebra, local rings play the same role as prime numbers in number theory: any local ring is indecomposable, i.e., it cannot be expressed as a product of two non-trivial rings. Also, for artinian rings, a counterpart of unique factorization holds: each artinian ring can be uniquely (up to isomorphism and order of factors) expressed as a product of local artinian rings.

It is therefore especially interesting to study properties of local rings from the computational point of view. Our research is motivated by work of Friedl, Rónyai, Eberly

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and others [7,14,2,8,3,4,9,5] who investigated computational aspects of finite dimensional algebras not only over finite fields but also over algebraic number or algebraic function fields. They, however, focus mainly on the (more general) problems of finding the Jacobson radical in a ring and then computing the simple components of the quotient by the radical. Their algorithms are efficient but randomized, it is therefore natural to ask whether the simpler problem of testing locality has a *deterministic* polynomial-time algorithm. An obvious way to test whether a commutative algebra A is local is to compute its Jacobson radical $J(A)$ and test whether the quotient algebra $A/J(A)$ is a field. In most interesting cases these tasks can be done in deterministic polynomial-time [7,14,2].

In this paper we propose another method based on our new characterization of local algebras. We focus on locality of commutative algebras with unit (and, unless stated otherwise, all algebras are supposed to be commutative with multiplicative identity), but we believe that our results can be applied to non-commutative setting as well. In [15], the second author presented a simple characterization of finite local algebras that leads to fast algorithms for testing locality of finite rings (here by "finite" we mean "with finitely many elements"). The characterization is based on the notion of a *local element* in an algebra: if A is an algebra over a field F, then an element $\alpha \in A$ is *local* if $F[\alpha]$ is a local subalgebra of A. The following has been proved in [15, Theorem 2.4]:

Theorem 1. *A finite algebra A over a (finite) field F is local if and only if it is generated (as a vector space) by its local elements.* □

Moreover, two conjectures have been stated, generalizing the characterization for an arbitrary field F:

Conjecture 2. *Let A be a finite-dimensional F-algebra. Then A is local if and only if its local elements generate A as a vector space over F.*

And more general

Conjecture 3. *Let A be an algebraic F-algebra. Then A is local if and only if its local elements generate A as a vector space over F.*

In this paper we settle Conjecture 2 in positive, thereby generalizing the characterization from [15]. Moreover, we settle Conjecture 3 in positive for base fields of characteristic 0. For positive characteristic, Conjecture 3 turns out to be false.

The paper is organized as follows. After the preliminary Section 2 we prove our main results in Section 3. In Section 4 we give sufficient conditions for an F-algebra to be local when its (algebraic) generators are local. As an application of the results from Section 3, in Section 5 we present and briefly discuss efficient polynomial-time algorithms for testing locality of finite-dimensional algebras over algebraic number fields.