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Some remarks on Gill's theorems on Young modules

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ABSTRACT

In a recent paper, [10], Gill, using methods of the modular representation theory of finite groups, describes some results on the tensor product of Young modules for symmetric groups. We here give an alternative approach using the polynomial representation theory of general linear groups and the Schur functor. The main result is a formula for the multiplicities of Young modules in a tensor product in terms of the characters of the simple polynomial modules for general linear groups. Our approach is also valid for Young modules for Hecke algebras of type A at roots of unity and here the formula involves the characters of the simple polynomial modules for quantised general linear groups.

1. Introduction

In [10], Gill considers the coefficients obtained by expressing the tensor product of Young modules, for a symmetric group, as a direct sum of Young modules: $Y(\lambda) \otimes Y(\mu) = \bigoplus_\nu y^\nu_{\lambda,\mu} Y(\nu)$. Working over a field of positive characteristic p, and using methods of the modular representation theory of finite groups, he shows in particular that $y^{p\nu}_{p\lambda,p\mu} = y^\nu_{\lambda,\mu}$, [10], Theorem 3.6. He also gives a lower bound for the Cartan invariant $c_{\lambda,\mu}$ of the Schur algebra in terms of the base p expansions of λ and μ, [10] Theorem 4.1. We here adopt a different approach, by first considering the corresponding problems in the category of

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polynomial representations and then applying the Schur functor, as in [11], Chapter 6, or [6], Section 2.1. Our main result, see Theorem 3.2, is a formula for the coefficients $y^{\nu}_{\lambda,\mu}$ in terms of what we call modular Kostka numbers, which describe the weight space multiplicities of the irreducible polynomial modules. From this we obtain the first of Gill's results mentioned above. We also obtain the lower bound for the Cartan numbers.

It is no more difficult, from our point of view, to work with quantised general linear groups and Hecke algebras, so we adopt this point of view from the outset. However, there is one interesting difference in the quantum case. Whereas the tensor product $Y(\lambda) \otimes Y(\mu)$ has a natural module structure in the classical case, we only assign the tensor product of Young modules for the Hecke algebra a meaning as a virtual module, and we give an example (in Section 6) to show that this is not represented by a direct sum of Young modules in general.

2. Preliminaries

2.1. We begin by introducing the usual notation for the combinatorics associated with polynomial representations, for the most part following Green, [11]. By a partition we mean an infinite sequence of weakly decreasing non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ with $\lambda_i = 0$ for $i \gg 0$. The length of the zero partition is 0 and a non-zero partition λ has length l if $\lambda_l = 0$ and $\lambda_i = 0$ for $i > l$. We identify the partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ with the finite sequence $(\lambda_1, \ldots, \lambda_n)$ if $\lambda_{n+1} = 0$. By the degree of a finite sequence of non-negative integers we mean the sum of its terms.

Fix a positive integer n and a non-negative integer r. We denote by $\Lambda(n)$ the set of n-tuples of non-negative integers. We denote by $\Lambda(n, r)$ set of elements of $\Lambda(n)$ of degree r. We denote by $\Lambda^+(n)$ the set of partitions of length at most n and denote by $\Lambda^+(n, r)$ the set of partitions of length at most n and degree r. We write $P(r)$ for the set of all partitions of r (so $P(r) = \Lambda^+(n, r)$, for $n \geq r$). Elements of $\Lambda(n)$ will be called (polynomial) weights and elements of $\Lambda^+(n)$ called dominant (polynomial) weights. The set $\Lambda(n, r)$ has a natural partial order, namely the dominance order: for $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ we have $\alpha \leq \beta$ if α and β have the same degree and $\sum_{j=1}^{i} \alpha_j \leq \sum_{j=1}^{i} \beta_j$, for all $1 \leq i \leq n$.

We write $\mathrm{Sym}(r)$ for the group of symmetries of $\{1, \ldots, r\}$. Then $W = \mathrm{Sym}(n)$ acts naturally on $\Lambda(n)$ by place permutation. The integral monoid ring of $\Lambda(n)$ will be identified with the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$. The natural action of W on $x_1, \ldots, x_n$ extends to an action on $\mathbb{Z}[x_1, \ldots, x_n]$ by ring automorphisms. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \Lambda(n)$ we set $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ and, as in [13], define the monomial symmetric function

$$m_\lambda = \sum_{\alpha \in W\lambda} x^\alpha$$

for $\lambda \in \Lambda^+(n)$.