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# Weil restriction of noncommutative motives

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A B S T R A C T

The Weil restriction functor, introduced in the late fifties, was recently extended by Karpenko to the category of Chow motives with integer coefficients. In this article we introduce the noncommutative (= NC) analogue of the Weil restriction functor, where schemes are replaced by dg algebras, and extend it to Kontsevich's categories of NC Chow motives and NC numerical motives. Instead of integer coefficients, we work more generally with coefficients in a binomial ring. Along the way, we extend Karpenko's functor to the classical category of numerical motives, and compare this extension with its NC analogue. As an application, we compute the (NC) Chow motive of the Weil restriction of every smooth projective scheme whose category of perfect complexes admits a full exceptional collection. Finally, in the case of central simple algebras, we describe explicitly the NC analogue of the Weil restriction functor using solely the degree of the field extension. This leads to a "categorification" of the classical corestriction homomorphism between Brauer groups.



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## 1. Introduction

### 1.1. Weil restriction

Given a finite separable field extension $l/k$, Weil [49] introduced in the late fifties the *Weil restriction functor*

$$\mathcal{R}_{l/k} : \mathrm{QProj}(l) \longrightarrow \mathrm{QProj}(k) \tag{1.1}$$

from quasi-projective $l$-schemes to quasi-projective $k$-schemes. This functor is nowadays an important tool in algebraic geometry and number theory; see Milne's work [34] on the Swinnerton–Dyer conjecture. Conceptually, (1.1) is the right adjoint of the base-change functor. Among other properties, it preserves smoothness, projectiveness, and it is moreover symmetric monoidal. Hence, it restricts to a $\otimes$-functor

$$\mathcal{R}_{l/k} : \mathrm{SmProj}(l) \longrightarrow \mathrm{SmProj}(k) \tag{1.2}$$

from smooth projective $l$-schemes to smooth projective $k$-schemes. At the beginning of the millennium, Karpenko [16] extended (1.2) to $\otimes$-functors

$$\begin{tikzcd} \mathrm{SmProj}(l)^{\mathrm{op}} \arrow[r, "\mathcal{R}_{l/k}"] \arrow[d, "M"'] & \mathrm{SmProj}(k)^{\mathrm{op}} \arrow[d, "M"] \\ \mathrm{Chow}(l) \arrow[r, "\mathcal{R}_{l/k}"'] & \mathrm{Chow}(k) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{SmProj}(l)^{\mathrm{op}} \arrow[r, "\mathcal{R}_{l/k}"] \arrow[d, "M^*"'] & \mathrm{SmProj}(k)^{\mathrm{op}} \arrow[d, "M^*"] \\ \mathrm{Chow}^*(l) \arrow[r, "\mathcal{R}^*_{l/k}"'] & \mathrm{Chow}^*(k) \end{tikzcd} \tag{1.3}$$

defined on the categories of Chow motives with integer coefficients; $\mathrm{Chow}^*(-)$ is constructed using correspondences of arbitrary codimension. These latter $\otimes$-functors, although well-defined, are not additive! Consequently, they are not the right adjoints of the corresponding base-change functors.

### 1.2. Noncommutative motives

In noncommutative algebraic geometry in the sense of Bondal, Drinfeld, Kaledin, Kapranov, Kontsevich, Orlov, Stafford, Van den Bergh, and others (see [4–7,9,14,19–21,39,40]), schemes are replaced by differential graded (= dg) algebras. A celebrated result, due to Bondal and Van den Bergh [7], asserts that for every quasi-compact quasi-separated scheme $X$ there exists a dg algebra $A_X$ whose derived category $\mathcal{D}(A_X)$ is equivalent to $\mathcal{D}(X)$; see Section 12. The dg algebra $A_X$ is unique up to Morita equivalence and reflects many of the properties of $X$. For example, $X$ is smooth (resp. proper) if and only if $A_X$ is smooth (resp. proper) in the sense of Kontsevich; see Section 4. Let $\mathrm{Dga}(k)$ be the category of dg $k$-algebras and $\mathrm{SpDga}(k)$ its full subcategory of smooth